KOMPLEKSNI BROJEVI (m©h)

imaginarna jedinica **i** je broj koji kvadriran daje – 1, $i^2 = -1$

$$i = \sqrt{-1}$$

potencije imaginarne jedinice

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ tj.\ i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i,\ i^{-1} = \frac{1}{i} = -i$$

Primjer $\quad i^{23} = \begin{bmatrix} 23:4=5 \\ 3 \end{bmatrix} = i^3 = -i$

imaginaran broj

imaginaran broj jednak je umnošku realnog broja b i imaginarne jedinice i, $b \cdot i$

algebarski (standardni) oblik kompleksnog broja

$$z = a + b \cdot i$$

a je realni dio kompleksnog broja, a = Re z, b je imaginarni dio kompleksnog broja, b = Im z

Primjeri $\quad z = 4 - 7i \Rightarrow \operatorname{Re} z = 4\ ,\ \operatorname{Im} z = -7 \qquad z = 1 - i \Rightarrow \operatorname{Re} z = 1\ ,\ \operatorname{Im} z = -1$

zbrajanje kompleksnih brojeva $z_1 = a + b \cdot i \quad , \quad z_2 = c + d \cdot i$

$$z_1 + z_2 = a + b \cdot i + c + d \cdot i = (a + c) + (b + d) \cdot i$$

Primjer $\quad z_1 = 5 - 8i\ ,\ z_2 = 2 + 4i \Rightarrow z_1 + z_2 = 5 - 8i + 2 + 4i = 7 - 4i$

svojstva zbrajanja

$z_1 + z_2 = z_2 + z_1$	zakon komutacije (zamjene)
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	zakon asocijacije (združivanja)

kompleksna nula $\quad z = 0 + 0 \cdot i = (0 , 0)$

kompleksna jedinica $\quad z = 1 + 0 \cdot i = (1 , 0)$

imaginarna jedinica $\quad z = 0 + 1 \cdot i = (0 , 1)$

oduzimanje kompleksnih brojeva $z_1 = a + b \cdot i \quad , \quad z_2 = c + d \cdot i$

$$z_1 - z_2 = a + b \cdot i - (c + d \cdot i) = (a - c) + (b - d) \cdot i$$

Primjer $\quad z_1 = -3 + 7i\ ,\ z_2 = -4 + 9i \Rightarrow z_1 - z_2 = -3 + 7i - (-4 + 9i) = -3 + 7i + 4 - 9i = 1 - 2i$

množenje kompleksnih brojeva $z_1 = a + b \cdot i \quad , \quad z_2 = c + d \cdot i$

$$z_1 \cdot z_2 = (a + b \cdot i) \cdot (c + d \cdot i) = (ac - bd) + (ad + bc) \cdot i$$

Primjer

$$z_1 = 2 + 3i\ ,\ z_2 = 4 - 2i \Rightarrow z_1 \cdot z_2 = (2 + 3i) \cdot (4 - 2i) = 8 - 4i + 12i - 6i^2 = 8 - 4i + 12i + 6 = 14 + 8i$$

svojstva množenja

$z_1 \cdot z_2 = z_2 \cdot z_1$	zakon komutacije (zamjene)
$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$	zakon asocijacije (združivanja)
$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$	zakon distribucije množenja prema zbrajanju

dijeljenje kompleksnih brojeva $z_1 = a + b \cdot i \quad , \quad z_2 = c + d \cdot i$

$$\frac{z_1}{z_2} = \frac{a + b \cdot i}{c + d \cdot i} \cdot \frac{c - d \cdot i}{c - d \cdot i} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i\ ,\ z_2 \neq 0$$

Primjer $\quad \left.\begin{aligned} z_1 &= 1 + 2i \\ z_2 &= 2 + 3i \end{aligned}\right\} \Rightarrow \frac{z_1}{z_2} = \frac{1 + 2i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{2 - 3i + 4i + 6}{2^2 + 3^2} = \frac{8 + i}{13} = \frac{8}{13} + \frac{1}{13} \cdot i$

recipročna vrijednost kompleksnog broja z = a + b · i

$$\frac{1}{z}=\frac{1}{a+b\cdot i}\cdot\frac{a-b\cdot i}{a-b\cdot i}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}\cdot i$$

Primjer $\frac{1}{1+i}=\frac{1}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-i}{1^2+1^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}\cdot i$

jednakost kompleksnih brojeva $z_1 = a + b \cdot i$, $z_2 = c + d \cdot i$

$$z_1=z_2 \iff a=c \text{ i } b=d \text{ ili } \operatorname{Re} z_1=\operatorname{Re} z_2 \text{ i } \operatorname{Im} z_1=\operatorname{Im} z_2$$

kompleksno konjugiran broj broja z = a + b · i

$$\bar{z}=a-b\cdot i$$

Primjeri $z=4-7i \Rightarrow \bar{z}=4+7i$, $z=-2+5i \Rightarrow \bar{z}=-2-5i$

svojstva kompleksnog konjugiranja

$$\overline{z_1 \pm z_2}=\bar{z}_1 \pm \bar{z}_2 \ , \ \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2} \ , \ \overline{\left(z^n\right)}=\left(\bar{z}\right)^n \ , \ \overline{z_1\cdot z_2}=\bar{z}_1\cdot\bar{z}_2 \ , \ z\cdot\bar{z}=\left(\operatorname{Re} z\right)^2+\left(\operatorname{Im} z\right)^2 \ , \ \bar{\bar{z}}=z$$

modul kompleksnog broja z = a + b · i

$$|z|=r=\rho=|a+b\cdot i|=\sqrt{a^2+b^2}=\sqrt{\left(\operatorname{Re} z\right)^2+\left(\operatorname{Im} z\right)^2}$$

Primjer $z=\sqrt{2}+\sqrt{7}\,i \Rightarrow |z|=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{7}\right)^2}=\sqrt{2+7}=\sqrt{9}=3$

svojstva modula kompleksnog broja

$$|z_1\cdot z_2|=|z_1|\cdot|z_2| \ , \ |z^n|=|z|^n \ , \ \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|} \ , \ |z_1+z_2|\le|z_1|+|z_2|$$

dijeljenje kompleksnog broja realnim brojem

$$\frac{a+b\cdot i}{c}=\frac{a}{c}+\frac{b}{c}\cdot i$$

Primjer $\frac{6-8i}{2}=\frac{6}{2}-\frac{8}{2}i=3-4i$

drugi korijen iz negativnog broja

$$x>0 \Rightarrow \sqrt{-x}=i\cdot\sqrt{x}$$

Primjer $\sqrt{-16}=\sqrt{16\cdot(-1)}=\sqrt{16}\cdot\sqrt{-1}=4i$

trigonometrijski oblik kompleksnog broja

$$z=r\cdot\left(\cos\varphi+i\cdot\sin\varphi\right)$$

$r = \rho = |z|$ je modul ili apsolutna vrijednost kompleksnog broja (udaljenost od ishodišta kompleksne ravnine), φ je argument kompleksnog broja za kojeg vrijedi $0 \le \varphi \le 2\pi$ (oznaka φ = arg z)

eksponencijalni oblik kompleksnog broja

$z=r\cdot e^{i\cdot\varphi}$, e je baza prirodnog logaritma, e = 2.718281828...

veza pet najpoznatijih brojeva u matematici

$$e^{i\cdot\pi}+1=0$$

operacije s kompleksnim brojevima u trigonometrijskom obliku

$$z_1=r_1\cdot\left(\cos\varphi_1+i\cdot\sin\varphi_1\right) \ , \ z_2=r_2\cdot\left(\cos\varphi_2+i\cdot\sin\varphi_2\right)$$

zbrajanje

$$z_1+z_2=\left(r_1\cdot\cos\varphi_1+r_2\cdot\cos\varphi_2\right)+\left(r_1\cdot\sin\varphi_1+r_2\cdot\sin\varphi_2\right)\cdot i$$

oduzimanje

$$z_1-z_2=\left(r_1\cdot\cos\varphi_1-r_2\cdot\cos\varphi_2\right)+\left(r_1\cdot\sin\varphi_1-r_2\cdot\sin\varphi_2\right)\cdot i$$

množenje

$$z_1\cdot z_2=r_1\cdot r_2\cdot\left[\cos\left(\varphi_1+\varphi_2\right)+i\cdot\sin\left(\varphi_1+\varphi_2\right)\right]\ ,\ z_1\cdot z_2=r_1\cdot r_2\cdot e^{i\cdot(\varphi_1+\varphi_2)}$$

Primjer

$$\left.\begin{array}{l} z_1=2\cdot\left(\cos 30^0+i\cdot\sin 30^0\right)\\ z_2=3\cdot\left(\cos 60^0+i\cdot\sin 60^0\right)\end{array}\right\}\Rightarrow z_1\cdot z_2=6\cdot\left[\cos 90^0+i\cdot\sin 90^0\right]=6\cdot\left[0+i\cdot 1\right]=6i$$

dijeljenje

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\cdot\left[\cos\left(\varphi_1-\varphi_2\right)+i\cdot\sin\left(\varphi_1-\varphi_2\right)\right]\ ,\ \frac{z_1}{z_2}=\frac{r_1}{r_2}\cdot e^{i\cdot(\varphi_1-\varphi_2)}$$

Primjer

$$\left.\begin{array}{l} z_1=4\cdot\left(\cos 70^0+i\cdot\sin 70^0\right)\\ z_2=2\cdot\left(\cos 10^0+i\cdot\sin 10^0\right)\end{array}\right\}\Rightarrow \frac{z_1}{z_2}=\frac{4}{2}\cdot\left[\cos 60^0+i\cdot\sin 60^0\right]=2\cdot\left[\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right]=1+\sqrt{3}\,i$$

potenciranje

$$z^n=r^n\cdot\left(\cos n\varphi+i\cdot\sin n\varphi\right)\ ,\ z^n=r^n\cdot e^{i\cdot n\varphi}$$

Primjer

$$z=2\cdot\left(\cos 10^0+i\cdot\sin 10^0\right)$$

$$z^6=2^6\cdot\left(\cos 6\cdot 10^0+i\cdot\sin 6\cdot 10^0\right)=64\cdot\left(\cos 60^0+i\cdot\sin 60^0\right)=64\cdot\left[\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right]=32+32\cdot\sqrt{3}\,i$$

korjenovanje

$$\sqrt[n]{z}=\sqrt[n]{r}\cdot\left(\cos\frac{\varphi+k\cdot 2\pi}{n}+i\cdot\sin\frac{\varphi+k\cdot 2\pi}{n}\right)\ ,\ k=0,\ 1,\ 2,\ 3,\ \dots,\ n-1$$

Primjer

$$z=64\cdot\left(\cos 60^0+i\cdot\sin 60^0\right)\ ,\ \sqrt[6]{z}=?$$

$n=6$

$k=0,\ 1,\ 2,\ 3,\ 4,\ 5$

$$k=0\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+0\cdot 360^0}{6}+i\cdot\sin\frac{60^0+0\cdot 360^0}{6}\right)=2\cdot\left(\cos 10^0+i\cdot\sin 10^0\right)$$

$$k=1\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+1\cdot 360^0}{6}+i\cdot\sin\frac{60^0+1\cdot 360^0}{6}\right)=2\cdot\left(\cos 70^0+i\cdot\sin 70^0\right)$$

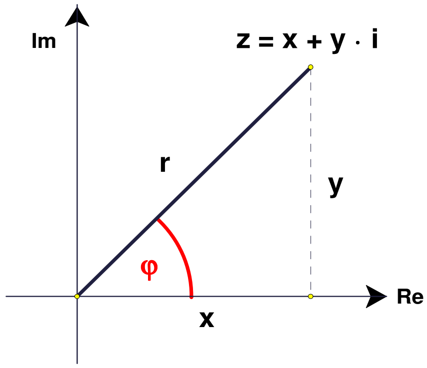
$$k=2\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+2\cdot 360^0}{6}+i\cdot\sin\frac{60^0+2\cdot 360^0}{6}\right)=2\cdot\left(\cos 130^0+i\cdot\sin 130^0\right)$$

$$k=3\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+3\cdot 360^0}{6}+i\cdot\sin\frac{60^0+3\cdot 360^0}{6}\right)=2\cdot\left(\cos 190^0+i\cdot\sin 190^0\right)$$

$$k=4\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+4\cdot 360^0}{6}+i\cdot\sin\frac{60^0+4\cdot 360^0}{6}\right)=2\cdot\left(\cos 250^0+i\cdot\sin 250^0\right)$$

$$k=5\Rightarrow\sqrt[6]{z}=\sqrt[6]{64}\cdot\left(\cos\frac{60^0+5\cdot 360^0}{6}+i\cdot\sin\frac{60^0+5\cdot 360^0}{6}\right)=2\cdot\left(\cos 310^0+i\cdot\sin 310^0\right)$$

trigonometrijski prikaz kompleksnog broja (I. kvadrant kompleksne ravnine)



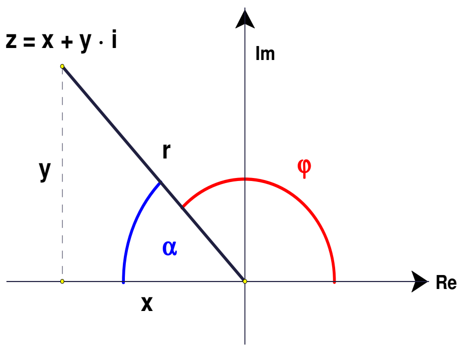
$$r = |z| = \sqrt{x^2 + y^2}$$

φ je argument kompleksnog broja z, $0 \le \varphi \le 2\pi$

$$tg\,\varphi = \frac{y}{x} \Rightarrow \varphi = arctg\,\frac{y}{x} = \tan^{-1}\frac{y}{x}$$

$$z = r \cdot (\cos\varphi + i \cdot \sin\varphi)$$

trigonometrijski prikaz kompleksnog broja (II. kvadrant kompleksne ravnine)



$$r = |z| = \sqrt{x^2 + y^2}$$

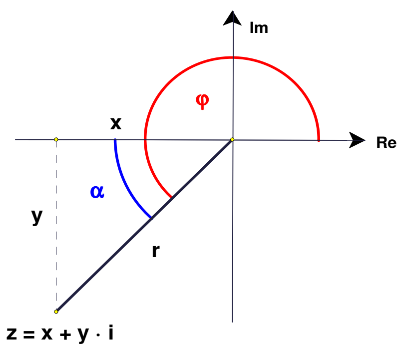
$$tg\,\alpha = \frac{y}{x} \Rightarrow \alpha = arctg\,\frac{y}{x} = \tan^{-1}\frac{y}{x}$$

φ je argument kompleksnog broja z, $0 \le \varphi \le 2\pi$

$$\varphi = 180^0 - \alpha \text{ ili } \varphi = \pi - \alpha$$

$$z = r \cdot (\cos\varphi + i \cdot \sin\varphi)$$

trigonometrijski prikaz kompleksnog broja (III. kvadrant kompleksne ravnine)



$$r = |z| = \sqrt{x^2 + y^2}$$

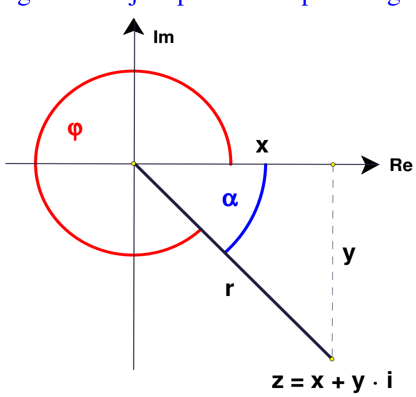
$$tg\,\alpha = \frac{y}{x} \Rightarrow \alpha = arctg\,\frac{y}{x} = \tan^{-1}\frac{y}{x}$$

φ je argument kompleksnog broja z, $0 \le \varphi \le 2\pi$

$$\varphi = 180^0 + \alpha \text{ ili } \varphi = \pi + \alpha$$

$$z = r \cdot (\cos\varphi + i \cdot \sin\varphi)$$

trigonometrijski prikaz kompleksnog broja (IV. kvadrant kompleksne ravnine)



$$r = |z| = \sqrt{x^2 + y^2}$$

$$tg\,\alpha = \frac{y}{x} \Rightarrow \alpha = arctg\,\frac{y}{x} = \tan^{-1}\frac{y}{x}$$

φ je argument kompleksnog broja z, $0 \le \varphi \le 2\pi$

$$\varphi = 360^0 - \alpha \text{ ili } \varphi = 2\pi - \alpha$$

$$z = r \cdot (\cos\varphi + i \cdot \sin\varphi)$$