Zadatak 081 (Mario, Ana, Mirjana, Mateo, srednja škola)

Kompleksan broj z = – 2 – 2 · i napišite u trigonometrijskom obliku.

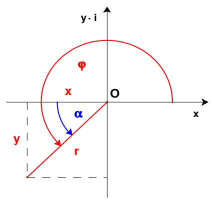
Rješenje 081

Ponovimo!

Kompleksan broj z = x + y · i napisan u trigonometrijskom obliku glasi

$$z = r \cdot (\cos \varphi + i \cdot \sin \varphi),$$

gdje je r apsolutna vrijednost ili modul kompleksnog broja (udaljenost kompleksnog broja od ishodišta kompleksne ravnine), φ argument kompleksnog broja.



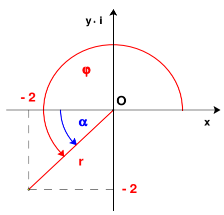
$$r = \sqrt{x^2 + y^2}$$

$$tg\,\alpha = \frac{y}{x} \Rightarrow \alpha = arctg\,\frac{y}{x} \text{ ili } tg\,\alpha = \frac{y}{x} \Rightarrow \alpha = tg^{-1}\left(\frac{y}{x}\right)$$

Argument φ iznosi:

$$\varphi = \pi + \alpha.$$

Ovaj kompleksan broj nalazi se u III. kvadrantu kompleksne ravnine, Gaussove ravnine, (x < 0, y < 0). Da bismo odredili njegov trigonometrijski oblik treba naći apsolutnu vrijednost r i argument φ:



$$\left.\begin{array}{l} z = -2 - 2 \cdot i \;,\; x = -2 \;,\; y = -2 \\ r = \sqrt{x^2 + y^2} \\ tg\,\alpha = \frac{y}{x} \\ \varphi = \pi + \alpha \end{array}\right\} \Rightarrow \left.\begin{array}{l} r = \sqrt{(-2)^2 + (-2)^2} \\ tg\,\alpha = \frac{2}{2} \\ \varphi = \pi + \alpha \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} r = \sqrt{4+4} \\ tg\,\alpha = 1 \\ \varphi = \pi + \alpha \end{array}\right\} \Rightarrow \left.\begin{array}{l} r = \sqrt{4 \cdot 2} \\ \alpha = arctg\,1 \\ \varphi = \pi + \alpha \end{array}\right\} \Rightarrow \left.\begin{array}{l} r = 2 \cdot \sqrt{2} \\ \alpha = \frac{\pi}{4} \\ \varphi = \pi + \alpha \end{array}\right\} \Rightarrow \left.\begin{array}{l} r = 2 \cdot \sqrt{2} \\ \varphi = \pi + \frac{\pi}{4} \end{array}\right\} \Rightarrow \left.\begin{array}{l} r = 2 \cdot \sqrt{2} \\ \varphi = \frac{5 \cdot \pi}{4} \end{array}\right\}.$$

Trigonometrijski oblik kompleksnog broja glasi:

$$z = 2 \cdot \sqrt{2} \cdot \left(\cos \frac{5 \cdot \pi}{4} + i \cdot \sin \frac{5 \cdot \pi}{4}\right).$$

Vježba 081

Kompleksan broj z = – 3 – 3 · i napišite u trigonometrijskom obliku.

Rezultat: $z = 3 \cdot \sqrt{2} \cdot \left(\cos \frac{5 \cdot \pi}{4} + i \cdot \sin \frac{5 \cdot \pi}{4}\right).$

Zadatak 082 (Mario, Ana, Mirjana, Mateo, srednja škola)

Kompleksan broj z = 2 – 2 · i napišite u trigonometrijskom obliku.

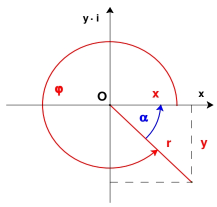
Rješenje 082

Ponovimo!

Kompleksan broj z = x + y · i napisan u trigonometrijskom obliku glasi

$$z = r \cdot (\cos \varphi + i \cdot \sin \varphi),$$

gdje je r apsolutna vrijednost ili modul kompleksnog broja (udaljenost kompleksnog broja od ishodišta kompleksne ravnine), φ argument kompleksnog broja.



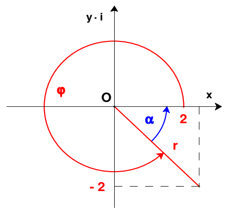
$$r=\sqrt{x^2+y^2}$$

$$tg\,\alpha=\frac{y}{x}\ \Rightarrow\ \alpha=arctg\,\frac{y}{x}\ \text{ ili }\ \mathrm{tg}\,\alpha=\frac{y}{x}\ \Rightarrow\ \alpha=tg^{-1}\left(\frac{y}{x}\right)$$

Argument φ iznosi:

$$\varphi=2\cdot\pi-\alpha.$$

Ovaj kompleksan broj nalazi se u IV. kvadrantu kompleksne ravnine, Gaussove ravnine, ($x > 0$, $y < 0$).
Da bismo odredili njegov trigonometrijski oblik treba naći apsolutnu vrijednost r i argument φ:



$$\left.\begin{matrix} z=2-2\cdot i\ ,\ x=2\ ,\ y=-2 \\ r=\sqrt{x^2+y^2} \\ tg\,\alpha=\dfrac{y}{x} \\ \varphi=2\cdot\pi-\alpha \end{matrix}\right\}\Rightarrow \left.\begin{matrix} r=\sqrt{2^2+(-2)^2} \\ tg\,\alpha=\dfrac{2}{2} \\ \varphi=2\cdot\pi-\alpha \end{matrix}\right\}\Rightarrow$$

$$\Rightarrow \left.\begin{matrix} r=\sqrt{4+4} \\ tg\,\alpha=1 \\ \varphi=2\cdot\pi-\alpha \end{matrix}\right\}\Rightarrow \left.\begin{matrix} r=\sqrt{4\cdot 2} \\ \alpha=arctg\,1 \\ \varphi=2\cdot\pi-\alpha \end{matrix}\right\}\Rightarrow \left.\begin{matrix} r=2\cdot\sqrt{2} \\ \alpha=\dfrac{\pi}{4} \\ \varphi=2\cdot\pi-\alpha \end{matrix}\right\}\Rightarrow \left.\begin{matrix} r=2\cdot\sqrt{2} \\ \varphi=2\cdot\pi-\dfrac{\pi}{4} \end{matrix}\right\}\Rightarrow \left.\begin{matrix} r=2\cdot\sqrt{2} \\ \varphi=\dfrac{7\cdot\pi}{4} \end{matrix}\right\}.$$

Trigonometrijski oblik kompleksnog broja glasi:

$$z=2\cdot\sqrt{2}\cdot\left(\cos\frac{7\cdot\pi}{4}+i\cdot\sin\frac{7\cdot\pi}{4}\right).$$

Vježba 082

Kompleksan broj $z = 3 - 3 \cdot i$ napišite u trigonometrijskom obliku.

Rezultat: $z=3\cdot\sqrt{2}\cdot\left(\cos\frac{7\cdot\pi}{4}+i\cdot\sin\frac{7\cdot\pi}{4}\right).$

Zadatak 083 (Martina, studentica)

Odredi sve kompleksne brojeve za koje vrijedi: $z^2+\dfrac{1+2\cdot i-\overline{(2-i)}}{1-(1+i)\cdot\overline{(1+i)}}=0.$

Rješenje 083

Ponovimo!

Neka je $z = x + y \cdot i$ bilo koji kompleksan broj. Sa $\bar{z}$ označavamo broj

$$\bar{z}=x-y\cdot i$$

koji nazivamo kompleksno konjugiranim broju z. Množeći z i $\bar{z}$ dobit ćemo:

$$(x+y\cdot i)\cdot(x-y\cdot i)=x^2+y^2.$$

Naći n – ti korijen kompleksnog broja

$$z=r\cdot(\cos\varphi+i\cdot\sin\varphi),$$

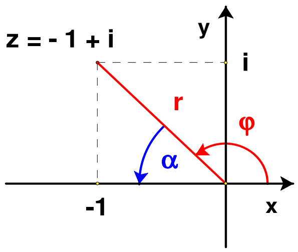
gdje je $n\in N$ i r modul kompleksnog broja, $r = |z|$, znači naći kompleksan broj

$$\sqrt[n]{z}=\sqrt[n]{r}\cdot\left(\cos\frac{\varphi+k\cdot 2\cdot\pi}{n}+i\cdot\sin\frac{\varphi+k\cdot 2\cdot\pi}{n}\right),\ k=0,\,1,\,2,\,...,\,n-1.$$

$$z^2+\frac{1+2\cdot i-\overline{(2-i)}}{1-(1+i)\cdot\overline{(1+i)}}=0 \Rightarrow z^2+\frac{1+2\cdot i-(2+i)}{1-(1+i)\cdot(1-i)}=0 \Rightarrow z^2+\frac{1+2\cdot i-2-i}{1-\left(1^2+1^2\right)}=0 \Rightarrow z^2+\frac{-1+i}{1-(1+1)}=0 \Rightarrow$$

$$\Rightarrow z^2+\frac{-1+i}{-1}=0 \Rightarrow z^2+1-i=0 \Rightarrow z^2=-1+i \;/\sqrt{\ } \Rightarrow z=\sqrt{-1+i}.$$

Da bismo našli $\sqrt{-1+i}$ moramo odrediti argument φ i modul r kompleksnog broja $z=-1+i$.



$$tg\,\alpha=\frac{y}{x} \Rightarrow tg\,\alpha=\frac{1}{1} \Rightarrow tg\,\alpha=1 \Rightarrow \alpha=arctg\,1 \Rightarrow \alpha=\frac{\pi}{4}.$$

Argument φ iznosi:

$$\varphi=\pi-\alpha \Rightarrow \varphi=\pi-\frac{\pi}{4} \Rightarrow \varphi=\frac{3\cdot\pi}{4}.$$

Modul r iznosi:

$$r=\sqrt{x^2+y^2} \Rightarrow r=\sqrt{(-1)^2+1^2} \Rightarrow r=\sqrt{1+1} \Rightarrow r=\sqrt{2}.$$

Zadatak ima dva rješenja:

- k = 0

$$\left.\begin{array}{l} n=2 \\ r=\sqrt{2}\ ,\ \varphi=\frac{3\cdot\pi}{4}\end{array}\right\} \Rightarrow \left(\sqrt{z}\right)_1=\sqrt{\sqrt{2}}\cdot\left(\cos\frac{\frac{3\cdot\pi}{4}+0\cdot2\cdot\pi}{2}+i\cdot\sin\frac{\frac{3\cdot\pi}{4}+0\cdot2\cdot\pi}{2}\right) \Rightarrow$$

$$\Rightarrow \left(\sqrt{z}\right)_1=\sqrt[4]{2}\cdot\left(\cos\frac{\frac{3\cdot\pi}{4}}{2}+i\cdot\sin\frac{\frac{3\cdot\pi}{4}}{2}\right) \Rightarrow \left(\sqrt{z}\right)_1=\sqrt[4]{2}\cdot\left(\cos\frac{3\cdot\pi}{8}+i\cdot\sin\frac{3\cdot\pi}{8}\right).$$

- k = 1

$$\left.\begin{array}{l} n=2 \\ r=\sqrt{2}\ ,\ \varphi=\frac{3\cdot\pi}{4}\end{array}\right\} \Rightarrow \left(\sqrt{z}\right)_2=\sqrt{\sqrt{2}}\cdot\left(\cos\frac{\frac{3\cdot\pi}{4}+1\cdot2\cdot\pi}{2}+i\cdot\sin\frac{\frac{3\cdot\pi}{4}+1\cdot2\cdot\pi}{2}\right) \Rightarrow$$

$$\Rightarrow \left(\sqrt{z}\right)_2=\sqrt[4]{2}\cdot\left(\cos\frac{\frac{3\cdot\pi}{4}+2\cdot\pi}{2}+i\cdot\sin\frac{\frac{3\cdot\pi}{4}+2\cdot\pi}{2}\right) \Rightarrow \left(\sqrt{z}\right)_2=\sqrt[4]{2}\cdot\left(\cos\frac{\frac{11\cdot\pi}{4}}{2}+i\cdot\sin\frac{\frac{11\cdot\pi}{4}}{2}\right) \Rightarrow$$

$$\Rightarrow \left(\sqrt{z}\right)_2=\sqrt[4]{2}\cdot\left(\cos\frac{11\cdot\pi}{8}+i\cdot\sin\frac{11\cdot\pi}{8}\right).$$

Vježba 083

Odredi sve kompleksne brojeve za koje vrijedi: $z^2+\frac{2+3\cdot i-\overline{(3-2\cdot i)}}{1-(1-i)\cdot\overline{(1-i)}}=0.$

Rezultat: $\left(\sqrt{z}\right)_1=\sqrt[4]{2}\cdot\left(\cos\frac{3\cdot\pi}{8}+i\cdot\sin\frac{3\cdot\pi}{8}\right)$, $\left(\sqrt{z}\right)_2=\sqrt[4]{2}\cdot\left(\cos\frac{11\cdot\pi}{8}+i\cdot\sin\frac{11\cdot\pi}{8}\right).$

Zadatak 084 (2A, TUPŠ)

Odredi realne brojeva x i y tako da vrijedi zadana jednakost:

$$x+y\cdot i-2=4\cdot x\cdot i+3\cdot y+3\cdot i.$$

Rješenje 084

Ponovimo!
Dva kompleksna broja $z_1 = a + b \cdot i$ i $z_2 = c + d \cdot i$ jednaki su ako su im jednaki realni dijelovi, a = c, i jednaki imaginarni dijelovi, b = d. Pišemo:

$$\left.\begin{matrix} z_1 = a + b \cdot i \\ z_2 = c + d \cdot i \end{matrix}\right\} \Rightarrow z_1 = z_2 \Leftrightarrow \left.\begin{matrix} a = c \\ b = d \end{matrix}\right\}.$$

1.inačica

$$x + y \cdot i - 2 = 4 \cdot x \cdot i + 3 \cdot y + 3 \cdot i \Rightarrow (x-2) + y \cdot i = 3 \cdot y + (4 \cdot x + 3) \cdot i.$$

$(x-2) + y \cdot i$	$3 \cdot y + (4 \cdot x + 3) \cdot i$
Realni dio je: $x - 2$ (uz njega ne stoji i)	Realni dio je: $3 \cdot y$ (uz njega ne stoji i)
Imaginarni dio je: y (uz njega stoji i)	Imaginarni dio je: $4 \cdot x + 3$ (uz njega stoji i)

Uporabom definicije jednakosti kompleksnih brojeva dobije se sustav od dvije linearne jednadžbe sa dvije nepoznanice:

$$\left.\begin{matrix} x - 2 = 3 \cdot y \\ y = 4 \cdot x + 3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -4 \cdot x + y = 3 \end{matrix}\right\} \Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -4 \cdot x + y = 3 \;/ \cdot 3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -12 \cdot x + 3 \cdot y = 9 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow -11 \cdot x = 11 \;/ : (-11) \Rightarrow x = -1 \Rightarrow \left.\begin{matrix} x = -1 \\ -4 \cdot x + y = 3 \end{matrix}\right\} \Rightarrow -4 \cdot (-1) + y = 3 \Rightarrow 4 + y = 3 \Rightarrow y = 3 - 4 \Rightarrow$$

$$\Rightarrow y = -1 \Rightarrow \left.\begin{matrix} x = -1 \\ y = -1 \end{matrix}\right\}.$$

2.inačica (rješenje učenice Martine, 2A)

$$x + y \cdot i - 2 = 4 \cdot x \cdot i + 3 \cdot y + 3 \cdot i \Rightarrow x + y \cdot i - 4 \cdot x \cdot i - 3 \cdot y = 2 + 3 \cdot i \Rightarrow (x - 3 \cdot y) + (y - 4 \cdot x) \cdot i = 2 + 3 \cdot i \Rightarrow$$

$$\left.\begin{matrix} x - 3 \cdot y = 2 \\ y - 4 \cdot x = 3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -4 \cdot x + y = 3 \end{matrix}\right\} \Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -4 \cdot x + y = 3 \;/ \cdot 3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x - 3 \cdot y = 2 \\ -12 \cdot x + 3 \cdot y = 9 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow -11 \cdot x = 11 \;/ : (-11) \Rightarrow x = -1 \Rightarrow \left.\begin{matrix} x = -1 \\ -4 \cdot x + y = 3 \end{matrix}\right\} \Rightarrow -4 \cdot (-1) + y = 3 \Rightarrow 4 + y = 3 \Rightarrow y = 3 - 4 \Rightarrow$$

$$\Rightarrow y = -1 \Rightarrow \left.\begin{matrix} x = -1 \\ y = -1 \end{matrix}\right\}.$$

Vježba 084

Odredi realne brojeva x i y tako da vrijedi zadana jednakost:

$$x + 3 + 2 \cdot y \cdot i = 8 + 4 \cdot i.$$

Rezultat: x = 5, y = 2.

Zadatak 085 (Iva, srednja škola)

Koliko je $\left(i^{-23} + i^{96}\right)^{-2}$?

Rješenje 085

Ponovimo!

$$\left(a^n\right)^m = a^{n \cdot m} \quad , \quad i^0 = 1 \quad , \quad i^2 = -1 \quad , \quad i^3 = -i \quad , \quad i^{-1} = -i \quad , \quad a^{-n} = \frac{1}{a^n}$$

$$(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2.$$

Izračunamo posebno svaku potenciju. Eksponent potencije dijelimo brojem 4 i gledamo ostatak (koji može biti 0, 1, 2 ili 3).

- $i^{-23} = \left(i^{-1}\right)^{23} = (-i)^{23} = -i^{23} = -i^{3} = -(-i) = i.$ $\qquad 23:4=5$, ostatak 3
- $i^{96} = i^{0} = 1.$ $\qquad 96:4=24$, ostatak 0

Sada je:

$$\left(i^{-23}+i^{96}\right)^{-2} = (i+1)^{-2} = (1+i)^{-2} = \frac{1}{(1+i)^2} = \frac{1}{1+2\cdot i+i^2} = \frac{1}{1+2\cdot i-1} = \frac{1}{2\cdot i} = \frac{1}{2\cdot i}\cdot\frac{-i}{-i} =$$

$$= \frac{-i}{-2\cdot i^2} = \frac{-i}{-2\cdot(-1)} = \frac{-i}{2} = -\frac{1}{2}\cdot i = -0.5\cdot i.$$

Vježba 085

Koliko je $\left(i^{-23}+i^{80}\right)^{-2}$?

Rezultat: $-0.5\cdot i.$

Zadatak 086 (Tajanstvena, gimnazija)

Nađi skup svih točaka kompleksne ravnine za koje je $|z| = |z-1-i|$.

Rješenje 086

Ponovimo!

$$z = x+y\cdot i \Rightarrow |z| = \sqrt{x^2+y^2} \quad , \quad (a-b)^2 = a^2-2\cdot a\cdot b+b^2.$$

Napišimo kompleksan broj z u standardnom ili algebarskom obliku: z = x + y · i. Tada slijedi:

$$\left.\begin{array}{l} |z| = |z-1-i| \\ z = x+y\cdot i \end{array}\right\} \Rightarrow |x+y\cdot i| = |x+y\cdot i-1-i| \Rightarrow |x+y\cdot i| = |(x-1)+(y-1)\cdot i| \Rightarrow$$

$$\Rightarrow \sqrt{x^2+y^2} = \sqrt{(x-1)^2+(y-1)^2} \ /^2 \Rightarrow x^2+y^2 = (x-1)^2+(y-1)^2 \Rightarrow$$

$$\Rightarrow x^2+y^2 = x^2-2\cdot x+1+y^2-2\cdot y+1 \Rightarrow x^2+y^2 = x^2-2\cdot x+1+y^2-2\cdot y+1 \Rightarrow$$

$$\Rightarrow 0 = -2\cdot x-2\cdot y+2 \ /:2 \Rightarrow 0 = -x-y+1 \Rightarrow \left.\begin{array}{ll} y = -x+1 & \text{eksplicitni oblik jednadžbe pravca} \\ x+y-1 = 0 & \text{implicitni oblik jednadžbe pravca} \end{array}\right\}.$$

Vježba 086

Nađi skup svih točaka kompleksne ravnine za koje je $|z| = |z-1|$.

Rezultat: $x = \frac{1}{2}.$

Zadatak 087 (2A, TUPŠ)

Koliki je imaginarni dio broja $z = \frac{(1+i)^{1997}}{(1-i)^{1996}}$?

Rješenje 087

Ponovimo!

$$a^n\cdot a^m = a^{n+m} \quad , \quad \left(a^n\right)^m = a^{n\cdot m} \quad , \quad i^0 = 1 \quad , \quad i^2 = -1 \quad , \quad (-a)^{2\cdot n} = a^{2\cdot n}$$

$$(a+b)^2 = a^2 + 2\cdot a\cdot b + b^2 \quad , \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad , \quad z = x + y\cdot i \Rightarrow \operatorname{Im} z = y.$$

1.inačica

$$z = \frac{(1+i)^{1997}}{(1-i)^{1996}} = \frac{(1+i)^{1996}\cdot(1+i)}{(1-i)^{1996}} = \frac{(1+i)^{1996}}{(1-i)^{1996}}\cdot(1+i) = \left(\frac{1+i}{1-i}\right)^{1996}\cdot(1+i) = \left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^{1996}\cdot(1+i) =$$

$$= \left(\frac{(1+i)^2}{1^2+1^2}\right)^{1996}\cdot(1+i) = \left(\frac{1+2\cdot i+i^2}{1+1}\right)^{1996}\cdot(1+i) = \left(\frac{1+2\cdot i-1}{2}\right)^{1996}\cdot(1+i) = \left(\frac{2\cdot i}{2}\right)^{1996}\cdot(1+i) =$$

$$= i^{1996}\cdot(1+i) = \begin{bmatrix} 1996:4=499 \\ 0 \end{bmatrix} = i^0\cdot(1+i) = 1\cdot(1+i) = 1+i \Rightarrow \operatorname{Im} z = 1.$$

2.inačica

$$z = \frac{(1+i)^{1997}}{(1-i)^{1996}} = \frac{(1+i)^{1996}\cdot(1+i)}{(1-i)^{1996}} = \frac{\left((1+i)^2\right)^{998}\cdot(1+i)}{\left((1-i)^2\right)^{998}} = \frac{\left(1+2\cdot i+i^2\right)^{998}\cdot(1+i)}{\left(1-2\cdot i+i^2\right)^{998}} = \frac{(1+2\cdot i-1)^{998}\cdot(1+i)}{(1-2\cdot i-1)^{998}} =$$

$$= \frac{(2\cdot i)^{998}\cdot(1+i)}{(-2\cdot i)^{998}} = \frac{(2\cdot i)^{998}\cdot(1+i)}{(2\cdot i)^{998}} = 1+i \Rightarrow \operatorname{Im} z = 1.$$

Vježba 087

Koliki je realni dio broja $z = \frac{(1+i)^{1997}}{(1-i)^{1996}}$?

Rezultat: 1.

Zadatak 088 (Ines, gimnazija)

Koliko je $(1-i)\cdot(1+i)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right)$?

Rješenje 088

Ponovimo!

$$(a-b)\cdot(a+b) = a^2 - b^2 \quad , \quad i^0 = 1 \quad , \quad i^2 = -1.$$

1.inačica

$$(1-i)\cdot(1+i)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) = \underbrace{(1-i)\cdot(1+i)}_{\text{razlika kvadrata}}\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) =$$

$$= \left(1-i^2\right)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) = \underbrace{\left(1-i^2\right)\cdot\left(1+i^2\right)}_{\text{razlika kvadrata}}\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) =$$

$$= \left(1-i^4\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) = \underbrace{\left(1-i^4\right)\cdot\left(1+i^4\right)}_{\text{razlika kvadrata}}\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) = \left(1-i^8\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right) =$$

$$= \underbrace{\left(1-i^8\right)\cdot\left(1+i^8\right)}_{\text{razlika kvadrata}}\cdot\left(1+i^{16}\right) = \left(1-i^{16}\right)\cdot\left(1+i^{16}\right) = \underbrace{\left(1-i^{16}\right)\cdot\left(1+i^{16}\right)}_{\text{razlika kvadrata}} = 1-i^{32} = \begin{bmatrix} 32:4=8 \\ 0 \end{bmatrix} = 1-i^0 = 1-1 = 0.$$

2.inačica

$$(1-i)\cdot(1+i)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right)=(1-i)\cdot(1+i)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right)=$$

$$=(1-i)\cdot(1+i)\cdot(1-1)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right)=(1-i)\cdot(1+i)\cdot 0\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)\cdot\left(1+i^{16}\right)=0.$$

Vježba 088

Koliko je $(1-i)\cdot(1+i)\cdot\left(1+i^2\right)\cdot\left(1+i^4\right)\cdot\left(1+i^8\right)$?

Rezultat: 0.

Zadatak 089 (Goran, srednja škola)

Provjeri da za svaki prirodni broj n vrijedi: $\left(\frac{1-i}{\sqrt{2}}\right)^{2\cdot n}=(-i)^n$.

Rješenje 089

Ponovimo!

$$\left(a^n\right)^m=a^{n\cdot m} \quad , \quad (a-b)^2=a^2-2\cdot a\cdot b+b^2 \quad , \quad i^2=-1 \quad . \quad \left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$$

$$\left(\frac{1-i}{\sqrt{2}}\right)^{2\cdot n}=\left(\left(\frac{1-i}{\sqrt{2}}\right)^2\right)^n=\left(\frac{(1-i)^2}{\left(\sqrt{2}\right)^2}\right)^n=\left(\frac{1-2\cdot i+i^2}{2}\right)^n=\left(\frac{1-2\cdot i-1}{2}\right)^n=\left(\frac{-2\cdot i}{2}\right)^n=(-i)^n.$$

Vježba 089

Izračunaj: $\left(\frac{1-i}{\sqrt{2}}\right)^4$.

Rezultat: – 1.

Zadatak 090 (Anna, gimnazija)

Koji je skup točaka određen jednadžbom $|z+i|=4$?

Rješenje 090

Ponovimo!

$$z=x+y\cdot i \Rightarrow |z|=\sqrt{x^2+y^2} \quad , \quad \left.\begin{array}{l}(x-p)^2+(y-q)^2=r^2\\ S(p,q)-\text{središte kružnice}\\ r-\text{polumjer kružnice}\end{array}\right\}\text{jednadžba kružnice.}$$

Neka je $z_0 = x_0 + y_0 \cdot i$ bilo koji kompleksan broj. Skup $k=\left\{z : \left|z-z_0\right|=r\right\}$ skup je svih točaka z u kompleksnoj ravnini kojih je udaljenost do točke z_0 jednaka r. To je kružnica sa središtem z_0 i polumjerom r. Dakle, kružnica u kompleksnoj ravnini opisana je formulom

$$\left|z-z_0\right|=r.$$

$$|z+i|=4 \Rightarrow |z-(-i)|=4 \Rightarrow \left.\begin{array}{l}z_0=-i\\ r=4\end{array}\right\}.$$

Svi kompleksni brojevi z za koje vrijedi $|z+i|=4$ udaljeni su za 4 od broja $z_0 = -i$, dakle, leže na kružnici sa središtem u $z_0 = -i$ i polumjerom 4.

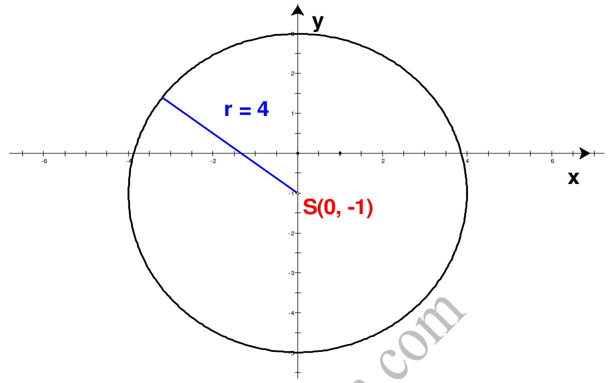
Računanjem apsolutne vrijednosti dobivamo njezinu jednadžbu u Kartezijevom sustavu. Napišemo kompleksan broj u standardnom ili algebarskom obliku $z = x + y \cdot i$:

$$|z+i|=4 \Rightarrow |x+y\cdot i+i|=4 \Rightarrow |x+(y+1)\cdot i|=4 \Rightarrow \sqrt{x^2+(y+1)^2}=4 \Rightarrow \left[\begin{array}{c}\text{kvadriramo}\\ \text{jednadžbu}\end{array}\right] \Rightarrow$$

$$\Rightarrow \sqrt{x^2+(y+1)^2}=4 \;/^2 \Rightarrow \left(\sqrt{x^2+(y+1)^2}\right)^2=4^2 \Rightarrow x^2+(y+1)^2=16 \Rightarrow$$

$$\left.\begin{array}{l}(x-p)^2+(y-q)^2=r^2\\ \Rightarrow x^2+(y+1)^2=16\\ p=0\ ,\ q=-1\ ,\ r^2=16 \Rightarrow r=4\end{array}\right\} \Rightarrow \left.\begin{array}{l}S(0,-1)\\ r=4\end{array}\right\}.$$

To je jednadžba kružnice sa središtem u točki S(0, – 1) i polumjerom r = 4.



Vježba 090

Koji je skup točaka određen jednadžbom $|z+i|=3$?

Rezultat: To je jednadžba kružnice sa središtem u točki S(0, – 1) i polumjerom r = 3.

Zadatak 091 (2A, TUPŠ)

U koordinatnoj ravnini predoči brojeve za koje je $|z+1|<|z-i|$.

Rješenje 091

Ponovimo!

$$z=x+y\cdot i \Rightarrow |z|=\sqrt{x^2+y^2} \quad , \quad (a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad (a-b)^2=a^2-2\cdot a\cdot b+b^2.$$

Kompleksan broj z napišemo u standardnom ili algebarskom obliku:

$$z = x + y \cdot i.$$

Tada je:

$$\left.\begin{array}{l}z=x+y\cdot i\\ |z+1|<|z-i|\end{array}\right\} \Rightarrow |x+y\cdot i+1|<|x+y\cdot i-i| \Rightarrow |(x+1)+y\cdot i|<|x+(y-1)\cdot i| \Rightarrow$$

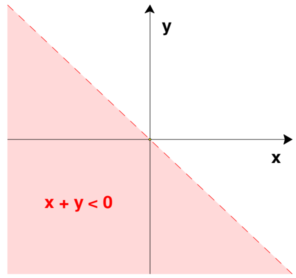
$$\Rightarrow \sqrt{(x+1)^2+y^2}<\sqrt{x^2+(y-1)^2} \Rightarrow \left[\begin{array}{l}\text{korijeni (korjenovi) su pozitivni pa prilikom}\\ \text{kvadriranja znak nejednakosti ne mijenja se}\end{array}\right] \Rightarrow$$

$$\Rightarrow \sqrt{(x+1)^2+y^2}<\sqrt{x^2+(y-1)^2} \;/^2 \Rightarrow \left(\sqrt{(x+1)^2+y^2}\right)^2<\left(\sqrt{x^2+(y-1)^2}\right)^2 \Rightarrow$$

$$\Rightarrow (x+1)^2+y^2<x^2+(y-1)^2 \Rightarrow x^2+2\cdot x+1+y^2<x^2+y^2-2\cdot y+1 \Rightarrow 2\cdot x<-2\cdot y \Rightarrow$$

$$\Rightarrow 2\cdot x+2\cdot y<0 \;/:2 \Rightarrow x+y<0 \Rightarrow y<-x. \quad [y=-x \text{ simetrala drugog i četvrtog kvadranta}]$$

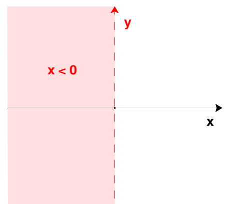
Grafički prikaz je poluravnina (crveno obojana) bez pravca y = – x.



Vježba 091

U koordinatnoj ravnini predoči brojeve za koje je $|z+1|<|z-1|$.

Rezultat: $x < 0$



Zadatak 092 (2A, TUPŠ)

U koordinatnoj ravnini predoči brojeve za koje je $|z+1|\le|z-i|$.

Rješenje 092

Ponovimo!

$$z=x+y\cdot i\;\Rightarrow\;|z|=\sqrt{x^2+y^2}\quad,\quad (a+b)^2=a^2+2\cdot a\cdot b+b^2\quad,\quad (a-b)^2=a^2-2\cdot a\cdot b+b^2.$$

Kompleksan broj z napišemo u standardnom ili algebarskom obliku:

$$z=x+y\cdot i.$$

Tada je:

$$\left.\begin{array}{l} z=x+y\cdot i \\ |z+1|\le|z-i| \end{array}\right\}\Rightarrow |x+y\cdot i+1|\le|x+y\cdot i-i|\Rightarrow|(x+1)+y\cdot i|\le|x+(y-1)\cdot i|\Rightarrow$$

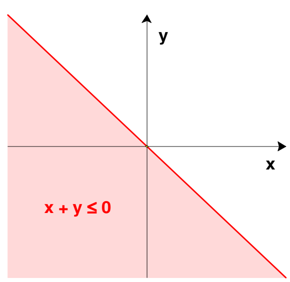
$$\Rightarrow\sqrt{(x+1)^2+y^2}\le\sqrt{x^2+(y-1)^2}\Rightarrow\left[\begin{array}{l}\text{korijeni(korjenovi) su pozitivni pa prilikom}\\ \text{kvadriranja znak nejednakosti ne mijenja se}\end{array}\right]\Rightarrow$$

$$\Rightarrow\sqrt{(x+1)^2+y^2}\le\sqrt{x^2+(y-1)^2}\;/^2\Rightarrow\left(\sqrt{(x+1)^2+y^2}\right)^2\le\left(\sqrt{x^2+(y-1)^2}\right)^2\Rightarrow$$

$$\Rightarrow(x+1)^2+y^2\le x^2+(y-1)^2\Rightarrow x^2+2\cdot x+1+y^2\le x^2+y^2-2\cdot y+1\Rightarrow 2\cdot x\le -2\cdot y\Rightarrow$$

$$\Rightarrow 2\cdot x+2\cdot y\le 0\;/:2\Rightarrow x+y\le 0\Rightarrow y\le -x.\quad[y=-x\text{ simetrala drugog i četvrtog kvadranta}]$$

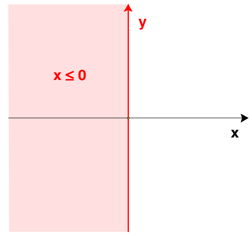
Grafički prikaz je poluravnina (crveno obojana) sa pravcem y = – x.



Vježba 092

U koordinatnoj ravnini predoči brojeve za koje je $|z+1|\le|z-1|$.

Rezultat: $x \le 0$



Zadatak 093 (2A, TUPŠ)

U koordinatnoj ravnini predoči brojeve za koje je $|z+1| = |z-i|$.

Rješenje 093

Ponovimo!

$$z = x + y \cdot i \Rightarrow |z| = \sqrt{x^2 + y^2} \quad , \quad (a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2 .$$

Kompleksan broj z napišemo u standardnom ili algebarskom obliku:

$$z = x + y \cdot i.$$

Tada je:

$$\left.\begin{array}{l} z = x + y \cdot i \\ |z+1| = |z-i| \end{array}\right\} \Rightarrow |x + y \cdot i + 1| = |x + y \cdot i - i| \Rightarrow |(x+1) + y \cdot i| = |x + (y-1) \cdot i| \Rightarrow$$

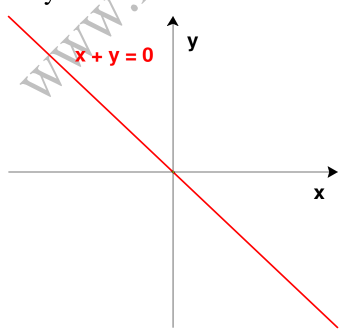
$$\Rightarrow \sqrt{(x+1)^2 + y^2} = \sqrt{x^2 + (y-1)^2} \Rightarrow \sqrt{(x+1)^2 + y^2} = \sqrt{x^2 + (y-1)^2} \;/^2 \Rightarrow$$

$$\Rightarrow \left(\sqrt{(x+1)^2 + y^2}\right)^2 = \left(\sqrt{x^2 + (y-1)^2}\right)^2 \Rightarrow$$

$$\Rightarrow (x+1)^2 + y^2 = x^2 + (y-1)^2 \Rightarrow x^2 + 2 \cdot x + 1 + y^2 = x^2 + y^2 - 2 \cdot y + 1 \Rightarrow 2 \cdot x = -2 \cdot y \Rightarrow$$

$$\Rightarrow 2 \cdot y = -2 \cdot x \;/:2 \Rightarrow y = -x. \text{ simetrala drugog i četvrtog kvadranta}$$

Grafički prikaz je pravac $y = -x$ ili $x + y = 0$.



Vježba 093

U koordinatnoj ravnini predoči brojeve za koje je $|z+1| = |z-1|$.

Rezultat: $x = 0$.

Zadatak 094 (2A, TUPŠ)

U koordinatnoj ravnini predoči brojeve za koje je $|z+1| > |z-i|$.

Rješenje 094

Ponovimo!

$$z = x + y \cdot i \Rightarrow |z| = \sqrt{x^2 + y^2} \quad , \quad (a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2 .$$

Kompleksan broj z napišemo u standardnom ili algebarskom obliku:

$$z = x + y \cdot i.$$

Tada je:

$$\left.\begin{array}{l} z = x + y \cdot i \\ |z+1| > |z-i| \end{array}\right\} \Rightarrow |x + y \cdot i + 1| > |x + y \cdot i - i| \Rightarrow |(x+1) + y \cdot i| > |x + (y-1) \cdot i| \Rightarrow$$

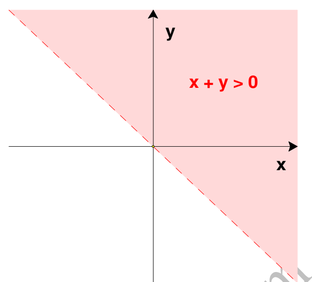
$$\Rightarrow \sqrt{(x+1)^2 + y^2} > \sqrt{x^2 + (y-1)^2} \Rightarrow \left[\begin{array}{l} \text{korijeni (korjenovi) su pozitivni pa prilikom} \\ \text{kvadriranja znak nejednakosti ne mijenja se} \end{array}\right] \Rightarrow$$

$$\Rightarrow \sqrt{(x+1)^2 + y^2} > \sqrt{x^2 + (y-1)^2} \; /^2 \Rightarrow \left(\sqrt{(x+1)^2 + y^2}\right)^2 > \left(\sqrt{x^2 + (y-1)^2}\right)^2 \Rightarrow$$

$$\Rightarrow (x+1)^2 + y^2 > x^2 + (y-1)^2 \Rightarrow x^2 + 2 \cdot x + 1 + y^2 > x^2 + y^2 - 2 \cdot y + 1 \Rightarrow 2 \cdot x > -2 \cdot y \Rightarrow$$

$$\Rightarrow 2 \cdot x + 2 \cdot y > 0 \; /:2 \Rightarrow x + y > 0 \Rightarrow y > -x. \quad [y = -x \text{ simetrala drugog i četvrtog kvadranta}]$$

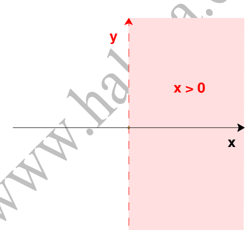
Grafički prikaz je poluravnina (crveno obojana) bez pravca y = – x.



Vježba 094

U koordinatnoj ravnini predoči brojeve za koje je | z + 1 | > | z – 1 |.

Rezultat: x > 0



Zadatak 095 (2A, TUPŠ)

U koordinatnoj ravnini predoči brojeve za koje je | z + 1 | ≥ | z – i |.

Rješenje 095

Ponovimo!

$$z = x + y \cdot i \Rightarrow |z| = \sqrt{x^2 + y^2} \quad , \quad (a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2.$$

Kompleksan broj z napišemo u standardnom ili algebarskom obliku:

z = x + y · i.

Tada je:

$$\left.\begin{array}{l} z = x + y \cdot i \\ |z+1| \ge |z-i| \end{array}\right\} \Rightarrow |x + y \cdot i + 1| \ge |x + y \cdot i - i| \Rightarrow |(x+1) + y \cdot i| \ge |x + (y-1) \cdot i| \Rightarrow$$

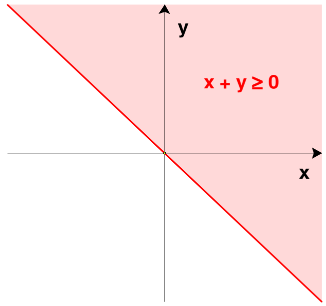
$$\Rightarrow \sqrt{(x+1)^2 + y^2} \ge \sqrt{x^2 + (y-1)^2} \Rightarrow \left[\begin{array}{l} \text{korijeni (korjenovi) su pozitivni pa prilikom} \\ \text{kvadriranja znak nejednakosti se mijenja se} \end{array}\right] \Rightarrow$$

$$\Rightarrow \sqrt{(x+1)^2 + y^2} \ge \sqrt{x^2 + (y-1)^2} \; /^2 \Rightarrow \left(\sqrt{(x+1)^2 + y^2}\right)^2 \ge \left(\sqrt{x^2 + (y-1)^2}\right)^2 \Rightarrow$$

$$\Rightarrow (x+1)^2 + y^2 \ge x^2 + (y-1)^2 \Rightarrow x^2 + 2 \cdot x + 1 + y^2 \ge x^2 + y^2 - 2 \cdot y + 1 \Rightarrow 2 \cdot x \ge -2 \cdot y \Rightarrow$$

$\Rightarrow\ 2\cdot x+2\cdot y\geq 0\ /:2\ \Rightarrow\ x+y\geq 0\ \Rightarrow\ y\geq -x.\quad [y=-x$ simetrala drugog i četvrtog kvadranta$]$

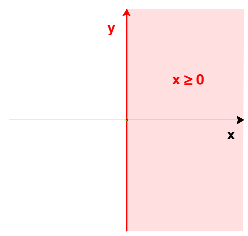
Grafički prikaz je poluravnina (crveno obojana) sa pravcem y = – x.



Vježba 095

U koordinatnoj ravnini predoči brojeve za koje je $|z+1|\geq|z-1|$.

Rezultat: $x\geq 0$



Zadatak 096 (Ivana, gimnazija)

Koliki je broj elemenata skupa $\left\{i^{k+1}+i^{-k-1}:k\in N\right\}$?

Rješenje 096

Ponovimo!

$\left(a^n\right)^m=a^{n\cdot m}\quad,\quad a^n\cdot a^m=a^{n+m}\quad,\quad i^0=1\quad,\quad i^1=i\quad,\quad i^2=-1\quad,\quad i^3=-i\quad,\quad i^{-1}=-i.$

$$i^{k+1}+i^{-k-1}=i^{k+1}+\left(i^{-1}\right)^{k+1}=i^{k+1}+(-i)^{k+1}=i^{k+1}+(-1\cdot i)^{k+1}=i^{k+1}+(-1)^{k+1}\cdot i^{k+1}=$$

$$=i^{k+1}\cdot\left(1+(-1)^{k+1}\right).$$

Računamo vrijednosti izraza za k = 1, 2, 3, 4, 5, 6, 7, 8, …:

$$\left.\begin{array}{l}k=1\\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\}\Rightarrow i^2\cdot\left(1+(-1)^2\right)=-1\cdot(1+1)=-1\cdot 2=-2,$$

$$\left.\begin{array}{l}k=2\\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\}\Rightarrow i^3\cdot\left(1+(-1)^3\right)=i^3\cdot(1-1)=i^3\cdot 0=0,$$

$$\left.\begin{array}{l}k=3\\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\}\Rightarrow i^4\cdot\left(1+(-1)^4\right)=\left[\begin{array}{l}4:4=1\\ 0\end{array}\right]=i^0\cdot\left(1+(-1)^4\right)=1\cdot(1+1)=1\cdot 2=2,$$

$$\left.\begin{array}{l}k=4\\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\}\Rightarrow i^5\cdot\left(1+(-1)^5\right)=i^5\cdot(1-1)=i^5\cdot 0=0,$$

$$\left.\begin{array}{l} k=5 \\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\} \Rightarrow i^{6}\cdot\left(1+(-1)^{6}\right)=\begin{bmatrix} 6:4=1 \\ 2 \end{bmatrix}=i^{2}\cdot\left(1+(-1)^{6}\right)=-1\cdot(1+1)=-1\cdot 2=-2,$$

$$\left.\begin{array}{l} k=6 \\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\} \Rightarrow i^{7}\cdot\left(1+(-1)^{7}\right)=i^{7}\cdot(1-1)=i^{7}\cdot 0=0,$$

$$\left.\begin{array}{l} k=7 \\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\} \Rightarrow i^{8}\cdot\left(1+(-1)^{8}\right)=\begin{bmatrix} 8:4=2 \\ 0 \end{bmatrix}=i^{0}\cdot(1+1)=1\cdot 2=2,$$

$$\left.\begin{array}{l} k=8 \\ i^{k+1}\cdot\left(1+(-1)^{k+1}\right)\end{array}\right\} \Rightarrow i^{9}\cdot\left(1+(-1)^{9}\right)=i^{9}\cdot(1-1)=i^{9}\cdot 0=0 \text{ itd.}$$

Elementi zadanog skupa su brojevi: – 2, 0, 2.

$$\left\{i^{k+1}+i^{-k-1} : k\in N\right\}=\{-2,\ 0,\ 2\}.$$

Broj elemnata je 3.

Vježba 096

Koliki je broj elemenata skupa $\left\{i^{k+1} : k\in N\right\}$?

Rezultat: 4.

Zadatak 097 (Tea, TUPŠ)

Ako je $z=\frac{1}{2}\cdot\left(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\cdot i\right)$, koliko iznosi z^{16}?

Rješenje 097

Ponovimo!

$$\left(a^{n}\right)^{m}=a^{n\cdot m} \quad , \quad (a\cdot b)^{n}=a^{n}\cdot b^{n} \quad , \quad (a+b)^{2}=a^{2}+2\cdot a\cdot b+b^{2} \quad , \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$$

$$i^{2}=-1 \quad , \quad (a+b)\cdot(a-b)=a^{2}-b^{2} \quad , \quad \sqrt{a^{2}}=a\ ,\ a\ge 0 \quad , \quad \left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}} \quad , \quad i^{0}=1.$$

$$z^{16}=\left(z^{2}\right)^{8}=\left(\left(\frac{1}{2}\cdot\left(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\cdot i\right)\right)^{2}\right)^{8}=\left(\left(\frac{1}{2}\right)^{2}\cdot\left(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}\cdot i\right)^{2}\right)^{8}=$$

$$=\left(\frac{1}{4}\cdot\left(\left(\sqrt{2+\sqrt{2}}\right)^{2}+2\cdot\sqrt{2+\sqrt{2}}\cdot\sqrt{2-\sqrt{2}}\cdot i+\left(\sqrt{2-\sqrt{2}}\cdot i\right)^{2}\right)\right)^{8}=$$

$$=\left(\frac{1}{4}\cdot\left(2+\sqrt{2}+2\cdot\sqrt{\left(2+\sqrt{2}\right)\cdot\left(2-2\sqrt{2}\right)}\cdot i+\left(\sqrt{2-\sqrt{2}}\right)^{2}\cdot i^{2}\right)\right)^{8}=$$

$$=\left(\frac{1}{4}\cdot\left(2+\sqrt{2}+2\cdot\sqrt{2^{2}-\left(\sqrt{2}\right)^{2}}\cdot i+\left(2-\sqrt{2}\right)\cdot(-1)\right)\right)^{8}=\left(\frac{1}{4}\cdot\left(2+\sqrt{2}+2\cdot\sqrt{4-2}\cdot i-2+\sqrt{2}\right)\right)^{8}=$$

$$=\left(\frac{1}{4}\cdot\left(2\cdot\sqrt{2}+2\cdot\sqrt{2}\cdot i\right)\right)^{8}=\left(\frac{1}{4}\cdot 2\cdot\sqrt{2}\cdot(1+i)\right)^{8}=\left(\frac{\sqrt{2}}{2}\cdot(1+i)\right)^{8}=\left(\left(\frac{\sqrt{2}}{2}\cdot(1+i)\right)^{2}\right)^{4}=$$

$$=\left(\left(\frac{\sqrt{2}}{2}\right)^2\cdot(1+i)^2\right)^4=\left(\frac{2}{4}\cdot\left(1+2\cdot i+i^2\right)\right)^4=\left(\frac{1}{2}\cdot(1+2\cdot i-1)\right)^4=\left(\frac{1}{2}\cdot 2\cdot i\right)^4=i^4=\begin{bmatrix}4:4=1\\0\end{bmatrix}=i^0=1.$$

Vježba 097

Ako je $z=\frac{1}{2}\cdot\left(\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}}\cdot i\right)$, koliko iznosi z^{16}?

Rezultat: 1.

Zadatak 098 (2A, TUPŠ)

Napišite 5 kao umnožak nekih dvaju kompleksnih brojeva kojima su i realni i imaginarni dijelovi različiti od 0.

Rješenje 098

Ponovimo!

Standardni ili algebarski oblik kompleksnog broja je oblika

$$z=x+y\cdot i,$$

gdje su x i y realni brojevi.

Za kompleksne brojeve $x+y\cdot i$ i $x-y\cdot i$ kažemo da su kompleksno konjugirani jedan drugome. Simbol konjugiranja jest povlaka iznad broja koji se konjugira:

$$z=x+y\cdot i\ \Rightarrow\ \bar{z}=x-y\cdot i.$$

Umnožak kompleksnog broja i njemu konjugiranog broja uvijek je realan broj:

$$z\cdot\bar{z}=(x+y\cdot i)\cdot(x-y\cdot i)=\begin{bmatrix}\text{razlika}\\\text{kvadrata}\end{bmatrix}=x^2-y^2\cdot i^2=\left[i^2=-1\right]=x^2+y^2.$$

Kružnica je skup svih točaka ravnine koje su jednako udaljene od jedne čvrste točke ravnine koju nazivamo središtem kružnice. Za kružnicu polumjera r sa središtem u ishodištu rabimo naziv središnja ili centralna kružnica:

$$x^2+y^2=r^2.$$

Iz uvjeta zadatka slijedi da se broj 5 može napisati na beskonačno mnogo načina. Na primjer,

$5=(1+2\cdot i)\cdot(1-2\cdot i).$

Dokaz: $(1+2\cdot i)\cdot(1-2\cdot i)=1^2+2^2=1+4=5.$

$5=(2+i)\cdot(2-i).$

Dokaz: $(2+i)\cdot(2-i)=2^2+1^2=4+1=5.$

$5=\left(\sqrt{2}+\sqrt{3}\cdot i\right)\cdot\left(\sqrt{2}-\sqrt{3}\cdot i\right).$

Dokaz: $\left(\sqrt{2}+\sqrt{3}\cdot i\right)\cdot\left(\sqrt{2}-\sqrt{3}\cdot i\right)=\left(\sqrt{2}\right)^2+\left(\sqrt{3}\right)^2=2+3=5.$

$5=\left(\sqrt{3}+\sqrt{2}\cdot i\right)\cdot\left(\sqrt{3}-\sqrt{2}\cdot i\right).$

Dokaz: $\left(\sqrt{3}+\sqrt{2}\cdot i\right)\cdot\left(\sqrt{3}-\sqrt{2}\cdot i\right)=\left(\sqrt{3}\right)^2+\left(\sqrt{2}\right)^2=3+2=5.$

$5=\left(\frac{\sqrt{17}}{2}+\frac{\sqrt{3}}{2}\cdot i\right)\cdot\left(\frac{\sqrt{17}}{2}-\frac{\sqrt{3}}{2}\cdot i\right).$

Dokaz: $\left(\frac{\sqrt{17}}{2}+\frac{\sqrt{3}}{2}\cdot i\right)\cdot\left(\frac{\sqrt{17}}{2}-\frac{\sqrt{3}}{2}\cdot i\right)=\left(\frac{\sqrt{17}}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=\frac{17}{4}+\frac{3}{4}=\frac{20}{4}=5.$

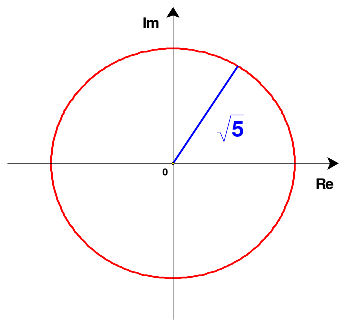
$$5=\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{14}}{2}\cdot i\right)\cdot\left(\frac{\sqrt{6}}{2}-\frac{\sqrt{14}}{2}\cdot i\right).$$

Dokaz: $\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{14}}{2}\cdot i\right)\cdot\left(\frac{\sqrt{6}}{2}-\frac{\sqrt{14}}{2}\cdot i\right)=\left(\frac{\sqrt{6}}{2}\right)^2+\left(\frac{\sqrt{14}}{2}\right)^2=\frac{6}{4}+\frac{14}{4}=\frac{20}{4}=5.$

Itd.

Uočimo da su rješenja sve točke koje pripadaju središnjoj kružnici polumjera $\sqrt{5}$:

$$x^2+y^2=\left(\sqrt{5}\right)^2 \Rightarrow x^2+y^2=5.$$



Vježba 098

Napišite 25 kao umnožak nekih dvaju kompleksnih brojeva kojima su i realni i imaginarni dijelovi različiti od 0.

Rezultat: $25 = (3 + 4 \cdot i) \cdot (3 - 4 \cdot i)$ itd.

Zadatak 099 (Alen, gimnazija)

Dokaži: $\sqrt{1+i\cdot\sqrt{3}}+\sqrt{1-i\cdot\sqrt{3}}=\sqrt{6}.$

Rješenje 099

Ponovimo!

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad (a+b)\cdot(a-b)=a^2-b^2 \quad , \quad \left(\sqrt{a}\right)^2=a \quad , \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}.$$

$$(a+b\cdot i)\cdot(a-b\cdot i)=a^2+b^2.$$

$$\sqrt{1+i\cdot\sqrt{3}}+\sqrt{1-i\cdot\sqrt{3}}=\sqrt{6} \Rightarrow \begin{bmatrix}\text{kvadriramo}\\ \text{jednakost}\end{bmatrix} \Rightarrow \sqrt{1+i\cdot\sqrt{3}}+\sqrt{1-i\cdot\sqrt{3}}=\sqrt{6} \;/^2 \Rightarrow$$

$$\Rightarrow \left(\sqrt{1+i\cdot\sqrt{3}}\right)^2+2\cdot\sqrt{1+i\cdot\sqrt{3}}\cdot\sqrt{1-i\cdot\sqrt{3}}+\left(\sqrt{1-i\cdot\sqrt{3}}\right)^2=\left(\sqrt{6}\right)^2 \Rightarrow$$

$$\Rightarrow 1+i\cdot\sqrt{3}+2\cdot\sqrt{\left(1+i\cdot\sqrt{3}\right)\cdot\left(1-i\cdot\sqrt{3}\right)}+1-i\cdot\sqrt{3}=6 \Rightarrow 2+2\cdot\sqrt{1+\left(\sqrt{3}\right)^2}=6 \Rightarrow$$

$$\Rightarrow 2+2\cdot\sqrt{1+3}=6 \Rightarrow 2+2\cdot\sqrt{4}=6 \Rightarrow 2+2\cdot 2=6 \Rightarrow 6=6.$$ Dokaz gotov.

Vježba 099

Dokaži: $\sqrt{1+i\cdot\sqrt{8}}+\sqrt{1-i\cdot\sqrt{8}}=\sqrt{8}.$

Rezultat: Tvrdnja točna.

Zadatak 100 (Alen, gimnazija)

Izračunaj: $(1-i)^2\cdot(1-2\cdot i)^2\cdot(1-3\cdot i)^2.$

Rješenje 100

Ponovimo!

$$a^n \cdot b^n = (a \cdot b)^n \quad , \quad i^2 = -1.$$

$$(1-i)^2 \cdot (1-2 \cdot i)^2 \cdot (1-3 \cdot i)^2 = \left((1-i) \cdot (1-2 \cdot i) \cdot (1-3 \cdot i)\right)^2 = \left(\left(1-2 \cdot i - i + 2 \cdot i^2\right) \cdot (1-3 \cdot i)\right)^2 =$$

$$= \left((1-2 \cdot i - i - 2) \cdot (1-3 \cdot i)\right)^2 = \left((-1-3 \cdot i) \cdot (1-3 \cdot i)\right)^2 = \left(-1+3 \cdot i - 3 \cdot i + 9 \cdot i^2\right)^2 =$$

$$= (-1 + 3 \cdot i - 3 \cdot i - 9)^2 = (-10)^2 = 100.$$

Vježba 100

Izračunaj: $(1-i)^4 \cdot (1-2 \cdot i)^4 \cdot (1-3 \cdot i)^4$.

Rezultat: 10000.