**Zadatak 081 (Ivana, studentica)**

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$.

**Rješenje 081**

Ponovimo!

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}\quad,\quad a^{-n}=\frac{1}{a^n}\quad,\quad \frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad \frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}\ ,\ \lim_{x\to a}g(x)\neq 0\quad,\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\quad,\quad \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n=e^k.$$

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x=\left[\begin{array}{l}\text{u brojniku i nazivniku}\\ \text{izlučimo } x\end{array}\right]=\lim_{x\to\infty}\left(\frac{x\cdot\left(1+\frac{1}{x}\right)}{x\cdot\left(1-\frac{1}{x}\right)}\right)^x=\lim_{x\to\infty}\left(\frac{x\cdot\left(1+\frac{1}{x}\right)}{x\cdot\left(1-\frac{1}{x}\right)}\right)^x=$$

$$=\lim_{x\to\infty}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)^x=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x}{\left(1-\frac{1}{x}\right)^x}=\frac{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x}{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x}=\frac{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x}{\lim_{x\to\infty}\left(1+\frac{-1}{x}\right)^x}=\frac{e}{e^{-1}}=e\cdot e=e^2.$$

**Vježba 081**

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x$.

**Rezultat:** $e^{-2}$.



**Zadatak 082 (Ivana, studentica)**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{x}{\sqrt{1+3\cdot x}-1}$.

**Rješenje 082**

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2\quad,\quad\left(\sqrt{a}\right)^2=a.$$

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3\cdot x}-1}=\left[\begin{array}{l}\text{racionalizacija}\\ \text{nazivnika}\end{array}\right]=\lim_{x\to 0}\frac{x}{\sqrt{1+3\cdot x}-1}\cdot\frac{\sqrt{1+3\cdot x}+1}{\sqrt{1+3\cdot x}+1}=$$

$$=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{\left(\sqrt{1+3\cdot x}-1\right)\cdot\left(\sqrt{1+3\cdot x}+1\right)}=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{\left(\sqrt{1+3\cdot x}\right)^2-1^2}=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{1+3\cdot x-1}=$$

$$=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{1+3\cdot x-1}=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{3\cdot x}=\lim_{x\to 0}\frac{x\cdot\left(\sqrt{1+3\cdot x}+1\right)}{3\cdot x}=\lim_{x\to 0}\frac{\sqrt{1+3\cdot x}+1}{3}=$$

$$=\frac{\sqrt{1+3\cdot 0}+1}{3}=\frac{\sqrt{1}+1}{3}=\frac{1+1}{3}=\frac{2}{3}.$$

**Vježba 082**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 0} \frac{\sqrt{1+3 \cdot x}-1}{x}$.

**Rezultat:** $\frac{3}{2}$.

**Zadatak 083 (Ivana, studentica)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 1} \left(\frac{3}{1-x^3}-\frac{1}{1-x}\right)$.

**Rješenje 083**

Ponovimo!

$$a^3-b^3=(a-b)\cdot\left(a^2+a\cdot b+b^2\right) \quad , \quad a^2-b^2=(a-b)\cdot(a+b).$$

$$\lim_{x \to 1}\left(\frac{3}{1-x^3}-\frac{1}{1-x}\right)=\lim_{x \to 1}\left(\frac{3}{(1-x)\cdot\left(1+x+x^2\right)}-\frac{1}{1-x}\right)=\lim_{x \to 1}\frac{3-1\cdot\left(1+x+x^2\right)}{(1-x)\cdot\left(1+x+x^2\right)}=$$

$$=\lim_{x \to 1}\frac{3-1-x-x^2}{(1-x)\cdot\left(1+x+x^2\right)}=\lim_{x \to 1}\frac{2-x-x^2}{(1-x)\cdot\left(1+x+x^2\right)}=\lim_{x \to 1}\frac{1-x+1-x^2}{(1-x)\cdot\left(1+x+x^2\right)}=$$

$$=\lim_{x \to 1}\frac{(1-x)+\left(1-x^2\right)}{(1-x)\cdot\left(1+x+x^2\right)}=\lim_{x \to 1}\frac{(1-x)+(1-x)\cdot(1+x)}{(1-x)\cdot\left(1+x+x^2\right)}=\left[\begin{array}{l}\text{u brojniku}\\ \text{izlučimo } 1-x\end{array}\right]=\lim_{x \to 1}\frac{(1-x)\cdot(1+1+x)}{(1-x)\cdot\left(1+x+x^2\right)}=$$

$$=\lim_{x \to 1}\frac{(1-x)\cdot(2+x)}{(1-x)\cdot\left(1+x+x^2\right)}=\lim_{x \to 1}\frac{(1-x)\cdot(2+x)}{(1-x)\cdot\left(1+x+x^2\right)}=\lim_{x \to 1}\frac{2+x}{1+x+x^2}=\frac{2+1}{1+1+1^2}=\frac{3}{1+1+1}=\frac{3}{3}=1.$$

**Vježba 083**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$.

**Rezultat:** – 1.

**Zadatak 084 (Ivana, studentica)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 0}\frac{\ln(1+x)}{x}$.

**Rješenje 084**

Ponovimo!

$$n \cdot \ln a=\ln a^n \quad , \quad \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n=e \quad , \quad \ln e=1.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x \to a} f(x)=f\left(\lim_{x \to a} x\right).$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\begin{bmatrix}\text{supstitucija}\\ \frac{1}{x}=t\ \Rightarrow\ x=\frac{1}{t}\\ x\to 0\ \Rightarrow\ t\to\infty\end{bmatrix}=$$

$$=\lim_{t\to\infty}\ln\left(1+\frac{1}{t}\right)^{t}=\ln\left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\right]=\ln e=1.$$

**Vježba 084**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\dfrac{x}{\ln(1+x)}$.

**Rezultat:** 1.

**Zadatak 085 (Ivana, studentica)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{x}$.

**Rješenje 085**

Ponovimo!

$$n\cdot\ln a=\ln a^{n}\quad,\quad \ln e^{n}=n\quad,\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e\quad,\quad \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}\quad,\quad \ln e=1.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a}f(x)=f\left(\lim_{x\to a}x\right).$$



$$\lim_{x\to 0}\frac{e^{x}-1}{x}=\begin{bmatrix}\text{supstitucija}\\ e^{x}-1=t\ \Rightarrow\ e^{x}=1+t\ \Rightarrow\ [\text{logaritmiramo}]\ \Rightarrow\\ \Rightarrow\ e^{x}=1+t\ /\ln\ \Rightarrow\ \ln e^{x}=\ln(1+t)\ \Rightarrow\ x=\ln(1+t)\\ x\to 0\ \Rightarrow\ t\to 0\end{bmatrix}=\lim_{t\to 0}\frac{t}{\ln(1+t)}=$$

$$=\lim_{t\to 0}\left[\frac{\ln(1+t)}{t}\right]^{-1}=\lim_{t\to 0}\left[\frac{1}{t}\cdot\ln(1+t)\right]^{-1}=\lim_{t\to 0}\left[\ln(1+t)^{\frac{1}{t}}\right]^{-1}=$$

$$=\begin{bmatrix}\text{supstitucija}\\ \frac{1}{y}=t\ \Rightarrow\ y=\frac{1}{t}\\ t\to 0\ \Rightarrow\ y\to\infty\end{bmatrix}=\lim_{y\to\infty}\left[\ln\left(1+\frac{1}{y}\right)^{y}\right]^{-1}=\left[\lim_{y\to\infty}\ln\left(1+\frac{1}{y}\right)^{y}\right]^{-1}=\left[\ln\left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]\right]^{-1}=$$

$$=(\ln e)^{-1}=1^{-1}=1.$$

**Vježba 085**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\dfrac{x}{e^{x}-1}$.

**Rezultat:** 1.

**Zadatak 086 (Marijan, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\left(\sqrt{n^{2}+2\cdot n}-n\right)$.

**Rješenje 086**

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2 \quad , \quad \left(\sqrt{a}\right)^2=a \quad , \quad \frac{a+b}{n}=\frac{a}{n}+\frac{b}{n} \quad , \quad \frac{\sqrt{a}}{b}=\sqrt{\frac{a}{b^2}}.$$

Ako su nizovi { $a_n$ } i { $b_n$ } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty} c=c \;,\; c-\text{konstanta} \quad , \quad \lim_{n\to\infty}\frac{c}{n}=0 \;,\; c-\text{konstanta} \quad , \quad \lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}(c\cdot a_n)=c\cdot\lim_{n\to\infty} a_n \;,\; c-\text{konstanta} \quad , \quad \lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty} a_n\pm\lim_{n\to\infty} b_n.$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty} a_n\cdot\lim_{n\to\infty} b_n \quad , \quad \lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \;,\; \lim_{n\to\infty} b_n\neq 0.$$

Pri traženju limesa količnika dvaju cijelih polinoma varijable n kada n → ∞, preporučljivo je oba člana količnika prethodno podijeliti s $n^p$, gdje je p najveći eksponent potencija tih polinoma.

$$\lim_{n\to\infty}\left(\sqrt{n^2+2\cdot n}-n\right)=\begin{bmatrix}\text{racionalizacija}\\ \text{brojnika}\end{bmatrix}=\lim_{n\to\infty}\left(\sqrt{n^2+2\cdot n}-n\right)\cdot\frac{\sqrt{n^2+2\cdot n}+n}{\sqrt{n^2+2\cdot n}+n}=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+2\cdot n}-n\right)\cdot\left(\sqrt{n^2+2\cdot n}+n\right)}{\sqrt{n^2+2\cdot n}+n}=\begin{bmatrix}\text{u brojniku je}\\ \text{razlika kvadrata}\end{bmatrix}=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+2\cdot n}\right)^2-n^2}{\sqrt{n^2+2\cdot n}+n}=$$

$$=\lim_{n\to\infty}\frac{n^2+2\cdot n-n^2}{\sqrt{n^2+2\cdot n}+n}=\lim_{n\to\infty}\frac{n^2+2\cdot n-n^2}{\sqrt{n^2+2\cdot n}+n}=\lim_{n\to\infty}\frac{2\cdot n}{\sqrt{n^2+2\cdot n}+n}=\begin{bmatrix}\text{brojnik i nazivnik}\\ \text{podijelimo sa } n\end{bmatrix}=$$

$$=\lim_{n\to\infty}\frac{\frac{2\cdot n}{n}}{\frac{\sqrt{n^2+2\cdot n}+n}{n}}=\lim_{n\to\infty}\frac{\frac{2\cdot n}{n}}{\frac{\sqrt{n^2+2\cdot n}}{n}+\frac{n}{n}}=\lim_{n\to\infty}\frac{2}{\frac{\sqrt{n^2+2\cdot n}}{n}+\frac{n}{n}}=\lim_{n\to\infty}\frac{2}{\frac{\sqrt{n^2+2\cdot n}}{n}+1}=$$

$$=\lim_{n\to\infty}\frac{2}{\sqrt{\frac{n^2+2\cdot n}{n^2}}+1}=\lim_{n\to\infty}\frac{2}{\sqrt{\frac{n^2}{n^2}+\frac{2\cdot n}{n^2}}+1}=\lim_{n\to\infty}\frac{2}{\sqrt{\frac{n^2}{n^2}+\frac{2\cdot n}{n^2}}+1}=\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{2}{n}}+1}=$$

$$=\frac{2}{\sqrt{1+0}+1}=\frac{2}{\sqrt{1}+1}=\frac{2}{1+1}=\frac{2}{2}=1.$$

**Vježba 086**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\left(\sqrt{n^2+3\cdot n}-n\right).$

**Rezultat:** $\frac{3}{2}.$

**Zadatak 087 (Marijan, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\frac{1^2+2^2+3^2+\ldots+n^2}{n^3}.$

**Rješenje 087**

Ponovimo!

Konačni red brojeva (konačna suma)

$$1^2+2^2+3^2+\ ...\ +(n-1)^2+n^2=\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}.$$

Ako su nizovi { $a_n$ } i { $b_n$ } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}\frac{c}{n}=0\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}(c\cdot a_n)=c\cdot\lim_{n\to\infty} a_n\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty} a_n\pm\lim_{n\to\infty} b_n.$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty} a_n\cdot\lim_{n\to\infty} b_n.$$

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\ ...\ +n^2}{n^3}=\lim_{n\to\infty}\frac{\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}}{n^3}=\lim_{n\to\infty}\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6\cdot n^3}=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{n^3}=\frac{1}{6}\cdot\lim_{n\to\infty}\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{n^3}=\frac{1}{6}\cdot\lim_{n\to\infty}\frac{(n+1)\cdot(2\cdot n+1)}{n^2}=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\frac{(n+1)\cdot(2\cdot n+1)}{n\cdot n}=\frac{1}{6}\cdot\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{2\cdot n+1}{n}=\frac{1}{6}\cdot\lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)\cdot\left(\frac{2\cdot n}{n}+\frac{1}{n}\right)=$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)\cdot\left(\frac{2\cdot n}{n}+\frac{1}{n}\right)=\frac{1}{6}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)=\frac{1}{6}\cdot(1+0)\cdot(2+0)=$$

$$=\frac{1}{6}\cdot1\cdot2=\frac{2}{6}=\frac{1}{3}.$$



**Vježba 087**

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\frac{3\cdot\left(1^2+2^2+3^2+\ ...\ +n^2\right)}{n^3}.$

**Rezultat:** 1.

**Zadatak 088 (Marijan, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\left(\sqrt{2\cdot n+3}-\sqrt{2\cdot n-1}\right).$

**Rješenje 088**

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2\quad,\quad\left(\sqrt{a}\right)^2=a\quad,\quad\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad\frac{\sqrt{a}}{b}=\sqrt{\frac{a}{b^2}}.$$

Ako su nizovi { $a_n$ } i { $b_n$ } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}\frac{c}{n}=0\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}(c\cdot a_n)=c\cdot\lim_{n\to\infty} a_n\ ,\ c-\text{konstanta}\quad,\quad \lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty} a_n\pm\lim_{n\to\infty} b_n.$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty} a_n\cdot\lim_{n\to\infty} b_n\quad,\quad\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}\ ,\ \lim_{n\to\infty} b_n\neq0.$$

Pri traženju limesa količnika dvaju cijelih polinoma varijable n kada $n\to\infty$, preporučljivo je oba člana količnika prethodno podijeliti s $n^p$, gdje je p najveći eksponent potencija tih polinoma.

1.inačica

$$\lim_{n\to\infty}\left(\sqrt{2\cdot n+3}-\sqrt{2\cdot n-1}\right)=\left[\begin{matrix}\text{racionalizacija}\\ \text{brojnika}\end{matrix}\right]=\lim_{n\to\infty}\left(\sqrt{2\cdot n+3}-\sqrt{2\cdot n-1}\right)\cdot\frac{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{2\cdot n+3}-\sqrt{2\cdot n-1}\right)\cdot\left(\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}\right)}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=\left[\begin{matrix}\text{u brojniku je}\\ \text{razlika kvadrata}\end{matrix}\right]=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{2\cdot n+3}\right)^2-\left(\sqrt{2\cdot n-1}\right)^2}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=\lim_{n\to\infty}\frac{2\cdot n+3-(2\cdot n-1)}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=\lim_{n\to\infty}\frac{2\cdot n+3-2\cdot n+1}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=$$

$$=\lim_{n\to\infty}\frac{2\cdot n+3-2\cdot n+1}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=\lim_{n\to\infty}\frac{4}{\sqrt{2\cdot n+3}+\sqrt{2\cdot n-1}}=0.$$

**Vježba 088**

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\left(\sqrt{2\cdot n+5}-\sqrt{2\cdot n-1}\right)$.

**Rezultat:** 0.

**Zadatak 089 (Dijana, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{2\cdot x-\sin x}{x}$.

**Rješenje 089**

Ponovimo!

$$\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}\quad,\quad \lim_{x\to 0}c=c\ ,\ c-\text{konstanta}\quad,\quad \lim_{x\to 0}\left(f(x)-g(x)\right)=\lim_{x\to 0}f(x)-\lim_{x\to 0}g(x).$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$



$$\lim_{x\to 0}\frac{2\cdot x-\sin x}{x}=\lim_{x\to 0}\left(\frac{2\cdot x}{x}-\frac{\sin x}{x}\right)=\lim_{x\to 0}\left(\frac{2\cdot x}{x}-\frac{\sin x}{x}\right)=\lim_{x\to 0}\left(2-\frac{\sin x}{x}\right)=$$

$$=\lim_{x\to 0}2-\lim_{x\to 0}\frac{\sin x}{x}=2-1=1.$$

**Vježba 089**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{3\cdot x-\sin x}{x}$.

**Rezultat:** 2.

**Zadatak 090 (Dijana, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{x^2-\sin x}{x}$.

**Rješenje 090**

Ponovimo!

$$\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}\quad,\quad \lim_{x\to 0}\left(f(x)-g(x)\right)=\lim_{x\to 0}f(x)-\lim_{x\to 0}g(x)\quad,\quad \lim_{x\to 0}\frac{\sin x}{x}=1.$$

$$\lim_{x\to 0}\frac{x^2-\sin x}{x}=\lim_{x\to 0}\left(\frac{x^2}{x}-\frac{\sin x}{x}\right)=\lim_{x\to 0}\left(\frac{x^2}{x}-\frac{\sin x}{x}\right)=\lim_{x\to 0}\left(x-\frac{\sin x}{x}\right)=$$

$$=\lim_{x\to 0} x-\lim_{x\to 0}\frac{\sin x}{x}=0-1=-1.$$

**Vježba 090**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{x^3-\sin x}{x}.$

**Rezultat:** $-1.$

**Zadatak 091 (Dijana, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{\sin ax}{x}.$

**Rješenje 091**

Ponovimo!

$$\lim_{x\to 0}(c\cdot f(x))=c\cdot\lim_{x\to 0} f(x)\ ,\ c-\text{konstanta}\quad ,\quad \lim_{x\to 0}\frac{\sin x}{x}=1.$$

$$\lim_{x\to 0}\frac{\sin ax}{x}=\begin{bmatrix}\text{supstitucija}\\ a\cdot x=u\ \Rightarrow\ x=\frac{u}{a}\\ x\to 0\ \Rightarrow\ u\to 0\end{bmatrix}=\lim_{u\to 0}\frac{\sin u}{\frac{u}{a}}=\lim_{u\to 0}\frac{a\cdot\sin u}{u}=a\cdot\lim_{u\to 0}\frac{\sin u}{u}=a\cdot 1=a.$$

**Vježba 091**

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{\sin 5x}{x}.$

**Rezultat:** 5.

**Zadatak 092 (Senad, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\left(1+\sqrt[n]{n}\right)\cdot\left(1+\frac{1}{n}\right)^{-2\cdot n}.$

**Rješenje 092**

Ponovimo!

Ako su nizovi $\{a_n\}$ i $\{b_n\}$ konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty} c=c\ ,\ c \text{ je konstanta}\quad ,\quad \lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n.$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n\quad ,\quad \lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}\ ,\ \lim_{n\to\infty}b_n\neq 0.$$

$$a^{-n}=\frac{1}{a^n}\quad ,\quad \lim_{n\to\infty}\sqrt[n]{n}=1\quad ,\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

$$\lim_{n\to\infty}\left(1+\sqrt[n]{n}\right)\cdot\left(1+\frac{1}{n}\right)^{-2\cdot n}=\lim_{n\to\infty}\frac{1+\sqrt[n]{n}}{\left(1+\frac{1}{n}\right)^{2\cdot n}}=\lim_{n\to\infty}\frac{1+\sqrt[n]{n}}{\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)^n}=$$

$$=\frac{\lim_{n\to\infty}\left(1+\sqrt[n]{n}\right)}{\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)^n\right)}=\frac{\lim_{n\to\infty}\left(1+\sqrt[n]{n}\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=$$

$$=\frac{\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\sqrt[n]{n}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}=\frac{1+\lim\limits_{n\to\infty}\sqrt[n]{n}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}=\frac{1+1}{e\cdot e}=\frac{2}{e^{2}}=2\cdot e^{-2}.$$

**Vježba 092**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\left(1+\sqrt[n]{n}\right)\cdot\left(1+\frac{1}{n}\right)^{2\cdot n}$.

**Rezultat:** $2\cdot e^{2}$.

**Zadatak 093 (Tomislav, student)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

**Rješenje 093**

Ponovimo!

| **Tablično deriviranje** | |
|---|---|
| **Funkcija** | **Derivacija** |
| ln x | $\frac{1}{x}$ |
| x | 1 |
| c, konstanta | 0 |
| $\frac{1}{x}$ | $-\frac{1}{x^{2}}$ |



Neka su f i g derivabilne funkcije na istom intervalu I.

Derivacija zbroja i razlike: $(f\pm g)'=f'\pm g'$.

Derivacija umnoška: $(f\cdot g)'=f'\cdot g+f\cdot g'$.

Derivacija umnoška konstantnog faktora i funkcije: (c · f(x)) ' = c · f ' (x)

Ponovimo L' Hospitalovo pravilo [čitaj: Lopitál].

Neka zadana funkcija ima oblik količnika $f(x)=\frac{u(x)}{v(x)}$ i neka je za x = a ona jednaka

$f(a)=\frac{u(a)}{v(a)}=\frac{0}{0}$ ili $f(a)=\frac{u(a)}{v(a)}=\frac{\infty}{\infty}$, tada je prava vrijednost funkcije f(x) za x = a, tj.

$$f(a)=\lim_{x\to a}\frac{u(x)}{v(x)}=\lim_{x\to a}\frac{u'(x)}{v'(x)}=\lim_{x\to a}\frac{u''(x)}{v''(x)}=\lim_{x\to a}\frac{u'''(x)}{v'''(x)}=\ldots\ \text{itd},$$

dok ne dođemo do određene vrijednosti konačne ili beskonačne. Derivira se posebno brojnik i posebno nazivnik, pri čemu se iza svakog deriviranja dobiveni izraz uredi i uvrsti zadani x = a. Zadatak je riješen, čim dobijemo u brojniku ili u nazivniku konačnu ili beskonačnu vrijednost.
Najprije oduzmemo razlomke u zagradi.

$$\lim_{x\to 1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1+}\frac{x\cdot\ln x-(x-1)}{(x-1)\cdot\ln x}=\lim_{x\to 1+}\frac{x\cdot\ln x-x+1}{(x-1)\cdot\ln x}.$$

Zatim primijenimo L' Hospitalovo pravilo.

$$\lim_{x\to 1+}\frac{x\cdot\ln x-x+1}{(x-1)\cdot\ln x}=\left[\begin{array}{c} x=1\ \Rightarrow\ \frac{1\cdot\ln 1-1+1}{(1-1)\cdot\ln 1}=\frac{1\cdot 0-1+1}{0\cdot 0}=\frac{0}{0} \\ \text{deriviramo posebno brojnik, posebno nazivnik}\end{array}\right]=$$

$$= \lim_{x\to 1+} \frac{(x\cdot\ln x - x+1)'}{((x-1)\cdot\ln x)'} = \lim_{x\to 1+}\frac{(x\cdot\ln x)' - x' + 1'}{(x-1)'\cdot\ln x + (x-1)\cdot(\ln x)'} = \lim_{x\to 1+}\frac{x'\cdot\ln x + x\cdot(\ln x)' - x' + 1'}{(x-1)'\cdot \ln x + (x-1)\cdot(\ln x)'} =$$

$$= \lim_{x\to 1+} \frac{1\cdot\ln x + x\cdot\frac{1}{x} - 1 + 0}{1\cdot\ln x + (x-1)\cdot\frac{1}{x}} = \lim_{x\to 1+}\frac{\ln x + 1 - 1 + 0}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1+} \frac{\ln x + {\color{magenta}1-1}+0}{\ln x+1-\frac{1}{x}} = \lim_{x\to 1+}\frac{\ln x}{\ln x + 1 - \frac{1}{x}} =$$

$$= \left[\begin{array}{c} x=1 \;\Rightarrow\; \dfrac{\ln 1}{\ln 1 + 1 - \frac{1}{1}} = \dfrac{0}{0+1-1} = {\color{magenta}\dfrac{0}{0}} \\ {\color{magenta}\text{deriviramo posebno brojnik, posebno nazivnik}} \end{array}\right] = \lim_{x\to 1+}\frac{(\ln x)'}{\left(\ln x + 1 - \frac{1}{x}\right)'} = \lim_{x\to 1+}\frac{(\ln x)'}{(\ln x)' + 1' - \left(\frac{1}{x}\right)'} =$$

$$= \lim_{x\to 1+}\frac{\frac{1}{x}}{\frac{1}{x} + 0 + \frac{1}{x^2}} = \lim_{x\to 1+}\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \left[\begin{array}{c} x=1 \;\Rightarrow\; \dfrac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{1^2}} = \dfrac{1}{\frac{1}{1}+\frac{1}{1}} \\ {\color{magenta}\text{zadatak je riješen}} \end{array}\right] = \frac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{1^2}} = \frac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{1}} = \frac{1}{1+1} = \frac{1}{2}.$$

**Vježba 093**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1+}\left(\frac{1}{\ln x} - \frac{x}{x-1}\right).$

**Rezultat:** $-\frac{1}{2}.$



**Zadatak 094 (Tonkica, studentica)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\frac{2-4\cdot x + x^2 + x^3}{x^5 - x^3 + x^2 - 1}.$

**Rješenje 094**

Ponovimo!

$${\color{magenta}a^2 - b^2 = (a-b)\cdot(a+b).}$$

$$\lim_{x\to 1}\frac{2-4\cdot x+x^2+x^3}{x^5-x^3+x^2-1} = \lim_{x\to 1}\frac{x^3+x^2-4\cdot x+2}{x^5-x^3+x^2-1} = \lim_{x\to 1}\frac{x^3+x^2{\color{magenta}-4\cdot x}+2}{x^5-x^3+x^2-1} =$$

$$= \lim_{x\to 1}\frac{x^3{\color{magenta}-x}+x^2{\color{magenta}-x-2\cdot x}+2}{x^5-x^3+x^2-1} = \lim_{x\to 1}\frac{x^3-x+x^2-x-2\cdot x+2}{x^5-x^3+x^2-1} = \lim_{x\to 1}\frac{\left(x^3-x\right)+\left(x^2-x\right)-(2\cdot x-2)}{\left(x^5-x^3\right)+\left(x^2-1\right)} =$$

$$= \lim_{x\to 1}\frac{x\cdot\left(x^2-1\right)+x\cdot(x-1)-2\cdot(x-1)}{x^3\cdot\left(x^2-1\right)+\left(x^2-1\right)} = \lim_{x\to 1}\frac{x\cdot(x-1)\cdot(x+1)+x\cdot(x-1)-2\cdot(x-1)}{x^3\cdot\left(x^2-1\right)+\left(x^2-1\right)} =$$

$$= \lim_{x\to 1}\frac{(x-1)\cdot(x\cdot(x+1)+x-2)}{\left(x^2-1\right)\cdot\left(x^3+1\right)} = \lim_{x\to 1}\frac{(x-1)\cdot\left(x^2+x+x-2\right)}{(x-1)\cdot(x+1)\cdot\left(x^3+1\right)} = \lim_{x\to 1}\frac{{\color{magenta}(x-1)}\cdot\left(x^2+2\cdot x-2\right)}{{\color{magenta}(x-1)}\cdot(x+1)\cdot\left(x^3+1\right)} =$$

$$=\lim_{x\to 1}\frac{x^2+2\cdot x-2}{(x+1)\cdot\left(x^3+1\right)}=\frac{1^2+2\cdot 1-2}{(1+1)\cdot\left(1^3+1\right)}=\frac{1+2-2}{2\cdot(1+1)}=\frac{1+2-2}{2\cdot 2}=\frac{1}{4}.$$

**Vježba 094**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\frac{x^3-2\cdot x+1}{x^3-2\cdot x^2+2\cdot x-1}$.

**Rezultat:** 1.

**Zadatak 095 (Vedran, srednja škola)**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\frac{2^n-3^n}{2^n+3^n}$.

**Rješenje 095**

Ponovimo!

$$\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}\quad,\quad\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}\quad,\quad|q|<1\Rightarrow\lim_{n\to\infty}|q|=0.$$

Ako su nizovi $\{a_n\}$ i $\{b_n\}$ konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty}c=c\ ,\ c\text{ je konstanta}\quad,\quad\lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n.$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}\ ,\ \lim_{n\to\infty}b_n\neq 0.$$



$$\lim_{n\to\infty}\frac{2^n-3^n}{2^n+3^n}=\lim_{n\to\infty}\frac{\frac{2^n-3^n}{3^n}}{\frac{2^n+3^n}{3^n}}=\lim_{n\to\infty}\frac{\frac{2^n-3^n}{3^n}}{\frac{2^n+3^n}{3^n}}=\lim_{n\to\infty}\frac{\frac{2^n}{3^n}-\frac{3^n}{3^n}}{\frac{2^n}{3^n}+\frac{3^n}{3^n}}=$$

$$=\lim_{n\to\infty}\frac{\left(\frac{2}{3}\right)^n-\frac{3^n}{3^n}}{\left(\frac{2}{3}\right)^n+\frac{3^n}{3^n}}=\lim_{n\to\infty}\frac{\left(\frac{2}{3}\right)^n-1}{\left(\frac{2}{3}\right)^n+1}=\frac{\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n-\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n+\lim\limits_{n\to\infty}1}=\frac{0-1}{0+1}=\frac{-1}{1}=-1.$$

**Vježba 095**

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\frac{2^n-5^n}{2^n+5^n}$.

**Rezultat:** – 1.

**Zadatak 096 (Hrvoje, student VTŠ)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 2}\frac{x^2-5\cdot x+6}{x-2}$.

**Rješenje 096**

Ponovimo!

Ako postoje $\lim\limits_{x\to a}f(x)$ i $\lim\limits_{x\to a}g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}c=c\ ,\ c\text{ je konstanta}\quad,\quad\lim_{x\to a}(f(x)\pm g(x)=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x).$$

Zakon distribucije množenja prema zbrajanju:

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

$$\lim_{x\to 2}\frac{x^2-5\cdot x+6}{x-2}=\lim_{x\to 2}\frac{x^2-5\cdot x+6}{x-2}=\lim_{x\to 2}\frac{x^2-2\cdot x-3\cdot x+6}{x-2}=\lim_{x\to 2}\frac{\left(x^2-2\cdot x\right)+(-3\cdot x+6)}{x-2}=$$

$$=\lim_{x\to 2}\frac{x\cdot(x-2)-3\cdot(x-2)}{x-2}=\left[\begin{array}{l}\text{u brojniku}\\ \text{izlučimo } x-2\end{array}\right]=\lim_{x\to 2}\frac{(x-2)\cdot(x-3)}{x-2}=\lim_{x\to 2}\frac{(x-2)\cdot(x-3)}{x-2}=$$

$$=\lim_{x\to 2}(x-3)=2-3=-1.$$

**Vježba 096**

Odredite graničnu vrijednost (limes): $\lim_{x\to 3}\frac{x^2-5\cdot x+6}{x-3}$.

**Rezultat:** 1.

## Zadatak 097 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$.

**Rješenje 097**

Ponovimo!

Ako postoje $\lim_{x\to a} f(x)$ i $\lim_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a} c=c \;,\; c \text{ je konstanta} \quad , \quad \lim_{x\to a}\left(f(x)\pm g(x)\right)=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x).$$

Zakon distribucije množenja prema zbrajanju:

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

$$\left(\sqrt{a}\right)^2=a \;,\; \left(\sqrt{a}\right)^4=a^2 \;,\; a^3-b^3=(a-b)\cdot\left(a^2+a\cdot b+b^2\right).$$



$$\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\left(\sqrt{x}\right)^4-\sqrt{x}}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\sqrt{x}\cdot\left(\left(\sqrt{x}\right)^3-1\right)}{\sqrt{x}-1}=$$

$$=\lim_{x\to 1}\frac{\sqrt{x}\cdot\left(\sqrt{x}-1\right)\cdot\left(\left(\sqrt{x}\right)^2+\sqrt{x}\cdot 1+1^2\right)}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\sqrt{x}\cdot\left(\sqrt{x}-1\right)\cdot\left(x+\sqrt{x}+1\right)}{\sqrt{x}-1}=$$

$$=\lim_{x\to 1}\frac{\sqrt{x}\cdot\left(\sqrt{x}-1\right)\cdot\left(x+\sqrt{x}+1\right)}{\sqrt{x}-1}=\lim_{x\to 1}\sqrt{x}\cdot\left(x+\sqrt{x}+1\right)=\sqrt{1}\cdot\left(1+\sqrt{1}+1\right)=1\cdot(1+1+1)=3.$$

**Vježba 097**

Odredite graničnu vrijednost (limes): $\lim_{x\to 1}\frac{x-\sqrt{x}}{\sqrt{x}-1}$.

**Rezultat:** 1.

## Zadatak 098 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{\sin 2x}{tg\, x}$.

**Rješenje 098**

Ponovimo!

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right).$$

$$\sin 2\alpha = 2 \cdot \sin\alpha \cdot \cos\alpha \quad , \quad tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} \quad , \quad \lim_{\alpha \to 0} \cos\alpha = 1.$$

$$\lim_{x \to 0} \frac{\sin 2x}{tg\,x} = \lim_{x \to 0} \frac{2 \cdot \sin x \cdot \cos x}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{\frac{2 \cdot \sin x \cdot \cos x}{1}}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{\frac{2 \cdot \sin x \cdot \cos x}{1}}{\frac{\sin x}{\cos x}} =$$

$$= \lim_{x \to 0} 2 \cdot \cos^2 x = 2 \cdot \lim_{x \to 0} \cos^2 x = 2 \cdot \left(\lim_{x \to 0} \cos x\right)^2 = 2 \cdot (\cos 0)^2 = 2 \cdot 1^2 = 2 \cdot 1 = 2.$$

**Vježba 098**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 0} ctg\,x \cdot \sin 2x.$

**Rezultat:** 2.

**Zadatak 099 (Hrvoje, student VTŠ)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \dfrac{x^2 + 7 \cdot x + 1}{3 - 2 \cdot x + (x-1)^2}.$



**Rješenje 099**

Ponovimo!

$$\frac{a-b}{n} = \frac{a}{n} - \frac{b}{n} \quad , \quad \frac{a+b}{n} = \frac{a}{n} + \frac{b}{n} \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2.$$

Ako su nizovi $\{a_n\}$ i $\{b_n\}$ konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n \to \infty} c = c \;,\; c \text{ je konstanta} \quad , \quad \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n.$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n \to \infty} a_n}{\lim\limits_{n \to \infty} b_n} \;,\; \lim_{n \to \infty} b_n \neq 0.$$

Pri traženju limesa količnika dvaju cijelih polinoma varijable x kada x → ∞, preporučljivo je oba člana količnika prethodno podijeliti s $x^p$, gdje je p najveći eksponent potencija tih polinoma.

Također je važno zapamtiti sljedeće limese

$$\lim_{x \to \infty} \frac{1}{x} = 0 \quad , \quad \lim_{x \to \infty} \frac{1}{x^2} = 0.$$

$$\lim_{x \to \infty} \frac{x^2 + 7 \cdot x + 1}{3 - 2 \cdot x + (x-1)^2} = \lim_{x \to \infty} \frac{x^2 + 7 \cdot x + 1}{3 - 2 \cdot x + x^2 - 2 \cdot x + 1} = \lim_{x \to \infty} \frac{x^2 + 7 \cdot x + 1}{x^2 - 4 \cdot x + 4} = \left[\begin{array}{l} \text{brojnik i nazivnik} \\ \text{dijelimo sa } x^2 \end{array}\right] =$$

$$=\lim_{x\to\infty}\frac{\frac{x^2+7\cdot x+1}{x^2}}{\frac{x^2-4\cdot x+4}{x^2}}=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{7\cdot x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{4\cdot x}{x^2}+\frac{4}{x^2}}=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{7\cdot x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{4\cdot x}{x^2}+\frac{4}{x^2}}=\lim_{x\to\infty}\frac{1+\frac{7}{x}+\frac{1}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}}=$$

$$=\frac{1+0+0}{1-0+0}=\frac{1}{1}=1.$$

**Vježba 099**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\frac{x^2+7\cdot x+1}{3-2\cdot x+(x+1)^2}$.

**Rezultat:** 1.

**Zadatak 100 (Hrvoje, student VTŠ)**

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\frac{x^2-3\cdot x+1}{(x-4)^2-(x+4)^2}$.

**Rješenje 100**

$$\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}\quad,\quad\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad(a-b)^2=a^2-2\cdot a\cdot b+b^2\quad,\quad(a+b)^2=a^2+2\cdot a\cdot b+b^2.$$

Ako su nizovi $\{a_n\}$ i $\{b_n\}$ konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty}c=c\ ,\ c\ \text{je konstanta}\quad,\quad\lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n.$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n},\ \lim_{n\to\infty}b_n\neq 0.$$



Pri traženju limesa količnika dvaju cijelih polinoma varijable x kada $x\to\infty$, preporučljivo je oba člana količnika prethodno podijeliti s $x^p$, gdje je p najveći eksponent potencija tih polinoma.

Također je važno zapamtiti sljedeće limese

$$\lim_{x\to\infty}\frac{1}{x}=0\quad,\quad\lim_{x\to\infty}\frac{1}{x^2}=0.$$

$$\lim_{x\to\infty}\frac{x^2-3\cdot x+1}{(x-4)^2-(x+4)^2}=\lim_{x\to\infty}\frac{x^2-3\cdot x+1}{x^2-8\cdot x+16-\left(x^2+8\cdot x+16\right)}=$$

$$=\lim_{x\to\infty}\frac{x^2-3\cdot x+1}{x^2-8\cdot x+16-x^2-8\cdot x-16}=\lim_{x\to\infty}\frac{x^2-3\cdot x+1}{x^2-8\cdot x+16-x^2-8\cdot x-16}=$$

$$=\lim_{x\to\infty}\frac{x^2-3\cdot x+1}{-16\cdot x}=\left[\begin{matrix}\text{brojnik i nazivnik}\\ \text{dijelimo sa } x^2\end{matrix}\right]=\lim_{x\to\infty}\frac{\frac{x^2-3\cdot x+1}{x^2}}{\frac{-16\cdot x}{x^2}}=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3\cdot x}{x^2}+\frac{1}{x^2}}{\frac{-16\cdot x}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3\cdot x}{x^2}+\frac{1}{x^2}}{\frac{-16\cdot x}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{1}{x^2}}{\frac{-16}{x}}=\frac{1-0+0}{0}=\frac{1}{0}=\infty.$$

**Vježba 100**

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{x^2 - 3 \cdot x + 2}{(x-4)^2 - (x+4)^2}$.

**Rezultat:** $\infty$.