Zadatak 041 (Ana, ekonomija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\sqrt{x\cdot(x+a)}-x\right).$

Rješenje 041

Ponovimo!

Određivanje limesa iracionalnog izraza često se radi tako da iracionalnost prebacimo iz brojnika u nazivnik ili obratno iz nazivnika u brojnik.

Razlika kvadrata: $(a-b)\cdot(a+b)=a^2-b^2.$

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}\quad,\quad\lim_{n\to\infty}\frac{1}{n}=0.$$

$$\lim_{x\to\infty}\left(\sqrt{x\cdot(x+a)}-x\right)=\left[\begin{array}{c}\text{pomnožimo s}\\ \dfrac{\sqrt{x\cdot(x+a)}+x}{\sqrt{x\cdot(x+a)}+x}\end{array}\right]=\lim_{x\to\infty}\left(\sqrt{x\cdot(x+a)}-x\right)\cdot\frac{\sqrt{x\cdot(x+a)}+x}{\sqrt{x\cdot(x+a)}+x}=$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x\cdot(x+a)}\right)^2-x^2}{\sqrt{x\cdot(x+a)}+x}=\lim_{x\to\infty}\frac{x\cdot(x+a)-x^2}{\sqrt{x^2+a\cdot x}+x}=\lim_{x\to\infty}\frac{x^2+a\cdot x-x^2}{\sqrt{x^2+a\cdot x}+x}=\lim_{x\to\infty}\frac{a\cdot x}{\sqrt{x^2+a\cdot x}+x}=$$

$$=\left[\begin{array}{c}\text{brojnik i nazivnik}\\ \text{podijelimo s } x\end{array}\right]=\lim_{x\to\infty}\frac{\dfrac{a\cdot x}{x}}{\dfrac{\sqrt{x^2+a\cdot x}}{x}+\dfrac{x}{x}}=\lim_{x\to\infty}\frac{a}{\sqrt{\dfrac{x^2+a\cdot x}{x^2}}+1}=\lim_{x\to\infty}\frac{a}{\sqrt{\dfrac{x^2}{x^2}+\dfrac{a\cdot x}{x^2}}+1}=$$

$$=\lim_{x\to\infty}\frac{a}{\sqrt{1+\dfrac{a}{x}}+1}=\frac{a}{\sqrt{1+0}+1}=\frac{a}{1+1}=\frac{a}{2}.$$

Vježba 041

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\sqrt{x\cdot(x+2)}-x\right).$

Rezultat: 1.

Zadatak 042 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x}}.$

Rješenje 042

Ponovimo!

Pri određivanju limesa oblika $\lim\limits_{x\to a}[\varphi(x)]^{\psi(x)}$ treba imati u vidu ovo:

- ako je $\lim\limits_{x\to a}\varphi(x)=1$ i $\lim\limits_{x\to a}\psi(x)=\infty$ onda stavljamo $\varphi(x)=1+\alpha(x)$, gdje je $\lim\limits_{x\to a}\alpha(x)=0$ i prema tome je:

$$\lim_{x\to a}[\varphi(x)]^{\psi(x)}=\lim_{x\to a}\left\{[1+\alpha(x)]^{\frac{1}{\alpha(x)}}\right\}^{\alpha(x)\cdot\psi(x)}=e^{\lim\limits_{x\to a}\alpha(x)\cdot\psi(x)}=$$

$$=e^{\lim\limits_{x\to a}[\varphi(x)-1]\cdot\psi(x)},$$ gdje je e = 2.718281828... Neperov broj.

Funkcija polovičnog kuta: $1-\cos x=2\cdot\sin^2\frac{x}{2}$.

Važni limesi: $\lim\limits_{n\to 0}\frac{\sin n}{n}=1$, $\lim\limits_{n\to 0}(1+n)^{\frac{1}{n}}=e$.

Potenciranje potencije: $a=\left(a^n\right)^{\frac{1}{n}}=\left(a^{\frac{1}{n}}\right)^n$, $a=\left(a^{\frac{b}{c}}\right)^{\frac{c}{b}}=\left(a^{\frac{c}{b}}\right)^{\frac{b}{c}}$.

Ako postoje $\lim\limits_{x\to a} f(x)$ i $\lim\limits_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x).$$

Računamo limes:

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}}=\lim_{x\to 0}\left[1-(1-\cos x)\right]^{\frac{1}{x}}=\lim_{x\to 0}\left[1-2\cdot\sin^2\frac{x}{2}\right]^{\frac{1}{x}}=$$

$$=\lim_{x\to 0}\left[\left(1-2\cdot\sin^2\frac{x}{2}\right)^{-\frac{1}{2\cdot\sin^2\frac{x}{2}}}\right]^{-\frac{2\cdot\sin^2\frac{x}{2}}{x}}=e^{\lim\limits_{x\to 0}\left(-\frac{2\cdot\sin^2\frac{x}{2}}{x}\right)}.$$

Posebno računamo $\lim\limits_{x\to 0}\left(-\frac{2\cdot\sin^2\frac{x}{2}}{x}\right)$.

1.inačica

$$\lim_{x\to 0}\left(-\frac{2\cdot\sin^2\frac{x}{2}}{x}\right)=-2\cdot\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{x}\right)=-2\cdot\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x}\cdot\sin\frac{x}{2}\right)=-2\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\cdot\lim_{x\to 0}\sin\frac{x}{2}=$$

$$=-\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\sin\frac{x}{2}=-1\cdot\lim_{x\to 0}\sin\frac{x}{2}=-1\cdot\sin 0=-1\cdot 0=0.$$

2.inačica

$$\lim_{x\to 0}\left(-\frac{2\cdot\sin^2\frac{x}{2}}{x}\right)=-2\cdot\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{x}\right)=-2\cdot\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x}\cdot\sin\frac{x}{2}\right)=$$

$$=-2\cdot\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{x}{2}\right)=-2\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{x}{2}=$$

$$=-\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{x}{2}=-1\cdot 1\cdot\lim_{x\to 0}\frac{x}{2}=-1\cdot 1\cdot 0=0.$$

Konačno rješenje glasi:

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}} = e^0 = 1.$$

Vježba 042

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

Rezultat: Naputak: $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = \lim_{x\to 0}\left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}} = e^{\lim_{x\to 0}\frac{\sin x}{x}} = e^1 = e.$

Zadatak 043 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{\ln(x+3)-\ln 3}{x}$.

Rješenje 043

Ponovimo!

Logaritam kvocijenta: $\log\frac{a}{b} = \log a - \log b \quad , \quad \ln\frac{a}{b} = \ln a - \ln b.$

Važan limes: $\lim_{n\to 0}(1+n)^{\frac{1}{n}} = e.$

Logaritam potencije: $\log a^n = n\cdot\log a \quad , \quad \ln a^n = n\cdot\ln a.$

Korisno je znati ukoliko egzistira i pozitivno je $\lim_{x\to a} f(x)$, da vrijedi:

$$\lim_{x\to a}(\ln f(x)) = \ln\left(\lim_{x\to a} f(x)\right).$$

Računamo limes:

$$\lim_{x\to 0}\frac{\ln(x+3)-\ln 3}{x} = \lim_{x\to 0}\frac{1}{x}\cdot\left[\ln(x+3)-\ln 3\right] = \lim_{x\to 0}\frac{1}{x}\cdot\ln\frac{x+3}{3} = \lim_{x\to 0}\ln\left(\frac{x+3}{3}\right)^{\frac{1}{x}} =$$

$$= \left[\ln\left(\lim_{x\to a} f(x)\right) = \lim_{x\to a}(\ln f(x))\right] = \ln\lim_{x\to 0}\left(\frac{x+3}{3}\right)^{\frac{1}{x}} = \ln\lim_{x\to 0}\left(1+\frac{x}{3}\right)^{\frac{1}{\frac{x}{3}\cdot 3}} =$$

$$= \ln\lim_{x\to 0}\left(\left(1+\frac{x}{3}\right)^{\frac{1}{\frac{x}{3}}}\right)^{\frac{1}{3}} = \ln e^{\frac{1}{3}} = \frac{1}{3}\cdot\ln e = \frac{1}{3}\cdot 1 = \frac{1}{3}.$$

Vježba 043

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{\ln(x+2)-\ln 2}{x}$.

Rezultat: $\frac{1}{2}$.

Zadatak 044 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\sqrt{x^2+b\cdot x+c}+\sqrt{x^2+d\cdot x+e}-2\cdot x\right).$

Rješenje 044

Ponovimo!

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Razlika kvadrata: $(a-b)\cdot(a+b)=a^2-b^2.$

Ako postoje $\lim\limits_{x\to a} f(x)$ i $\lim\limits_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x).$$

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}\quad,\quad\frac{\sqrt{a}}{b}=\sqrt{\frac{a}{b^2}}\quad,\quad\lim_{n\to\infty}\frac{1}{n}=0\quad,\quad\lim_{n\to\infty}\frac{1}{n^2}=0.$$

Računamo limes:

$$\lim_{x\to\infty}\left(\sqrt{x^2+b\cdot x+c}+\sqrt{x^2+d\cdot x+e}-2\cdot x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+b\cdot x+c}-x+\sqrt{x^2+d\cdot x+e}-x\right)=$$

$$=\lim_{x\to\infty}\left(\left(\sqrt{x^2+b\cdot x+c}-x\right)+\left(\sqrt{x^2+d\cdot x+e}-x\right)\right)=$$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+b\cdot x+c}-x\right)+\lim_{x\to\infty}\left(\sqrt{x^2+d\cdot x+e}-x\right)=$$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+b\cdot x+c}-x\right)\cdot\frac{\sqrt{x^2+b\cdot x+c}+x}{\sqrt{x^2+b\cdot x+c}+x}+\lim_{x\to\infty}\left(\sqrt{x^2+d\cdot x+e}-x\right)\cdot\frac{\sqrt{x^2+d\cdot x+e}+x}{\sqrt{x^2+d\cdot x+e}+x}=$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+b\cdot x+c}\right)^2-x^2}{\sqrt{x^2+b\cdot x+c}+x}+\lim_{x\to\infty}\frac{\left(\sqrt{x^2+d\cdot x+e}\right)^2-x^2}{\sqrt{x^2+d\cdot x+e}+x}=$$

$$=\lim_{x\to\infty}\frac{x^2+b\cdot x+c-x^2}{\sqrt{x^2+b\cdot x+c}+x}+\lim_{x\to\infty}\frac{x^2+d\cdot x+e-x^2}{\sqrt{x^2+d\cdot x+e}+x}=$$

$$=\lim_{x\to\infty}\frac{b\cdot x+c}{\sqrt{x^2+b\cdot x+c}+x}+\lim_{x\to\infty}\frac{d\cdot x+e}{\sqrt{x^2+d\cdot x+e}+x}=\left[\begin{array}{l}\text{brojnik i nazivnik svakog}\\ \text{razlomka dijelimo s } x\end{array}\right]=$$

$$=\lim_{x\to\infty}\frac{\frac{b\cdot x+c}{x}}{\frac{\sqrt{x^2+b\cdot x+c}+x}{x}}+\lim_{x\to\infty}\frac{\frac{d\cdot x+e}{x}}{\frac{\sqrt{x^2+d\cdot x+e}+x}{x}}=$$

$$= \lim_{x\to\infty} \frac{\frac{b\cdot x}{x}+\frac{c}{x}}{\frac{\sqrt{x^2+b\cdot x+c}}{x}+\frac{x}{x}} + \lim_{x\to\infty} \frac{\frac{d\cdot x}{x}+\frac{e}{x}}{\frac{\sqrt{x^2+d\cdot x+e}}{x}+\frac{x}{x}} =$$

$$= \lim_{x\to\infty} \frac{b+\frac{c}{x}}{\sqrt{\frac{x^2+b\cdot x+c}{x^2}}+1} + \lim_{x\to\infty} \frac{d+\frac{e}{x}}{\sqrt{\frac{x^2+d\cdot x+e}{x^2}}+1} =$$

$$= \lim_{x\to\infty} \frac{b+\frac{c}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{b\cdot x}{x^2}+\frac{c}{x^2}}+1} + \lim_{x\to\infty} \frac{d+\frac{e}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{d\cdot x}{x^2}+\frac{e}{x^2}}+1} =$$

$$= \lim_{x\to\infty} \frac{b+\frac{c}{x}}{\sqrt{1+\frac{b}{x}+\frac{c}{x^2}}+1} + \lim_{x\to\infty} \frac{d+\frac{e}{x}}{\sqrt{1+\frac{d}{x}+\frac{e}{x^2}}+1} = \left[\begin{array}{c} \text{važni limesi} \\ \lim_{n\to\infty} \frac{1}{n}=0 \;,\; \lim_{n\to\infty} \frac{1}{n^2}=0 \end{array}\right] =$$

$$= \frac{b+0}{\sqrt{1+0+0}+1} + \frac{d+0}{\sqrt{1+0+0}+1} = \frac{b}{\sqrt{1}+1} + \frac{d}{\sqrt{1}+1} = \frac{b}{2}+\frac{d}{2} = \frac{b+d}{2}.$$

Vježba 044

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty}\left(\sqrt{x^2+4\cdot x+1}+\sqrt{x^2+6\cdot x+1}-2\cdot x\right)$.

Rezultat: 5.

Zadatak 045 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\left(n\cdot\sin\frac{\pi}{n}\right)$.

Rješenje 045

Ponovimo!

Pri računanju limesa često se koristimo formulom: $\lim_{x\to 0}\frac{\sin x}{x}=1$.

$$\lim_{n\to\infty}\left(n\cdot\sin\frac{\pi}{n}\right) = \left[\begin{array}{l} \text{supstitucija (zamjena)} \\ x=\frac{\pi}{n} \;\Rightarrow\; n=\frac{\pi}{x} \\ n\to\infty \;\Rightarrow\; x\to 0 \end{array}\right] = \lim_{x\to 0}\left(\frac{\pi}{x}\cdot\sin x\right) = \lim_{x\to 0}\left(\pi\cdot\frac{\sin x}{x}\right) =$$

$$= \pi\cdot\lim_{x\to 0}\frac{\sin x}{x} = \pi\cdot 1 = \pi.$$

Vježba 045

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\left(n\cdot\sin\frac{2\pi}{n}\right)$.

Rezultat: 2π.

Zadatak 046 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\frac{x^3+x^2+2}{2\cdot x^3-x^2}\right)^{x^3}$.

Rješenje 046

Ponovimo!

Pri određivanju limesa oblika $\lim\limits_{x\to a}\left[\varphi(x)\right]^{\psi(x)}$ treba imati u vidu ovo:

- ako je $\lim\limits_{x\to a}\varphi(x)=A\neq 1$ i $\lim\limits_{x\to a}\psi(x)=\pm\infty$ onda pronalazimo zadani limes neposredno.

Važni limesi: $\lim\limits_{n\to\infty}\frac{1}{n}=0$, $\lim\limits_{n\to\infty}\frac{1}{n^3}=0$. Ako je 0 < a < 1, tada vrijedi: $\lim\limits_{n\to\infty}a^n=0$.

Računamo limes: $\lim\limits_{x\to\infty}\left(\frac{x^3+x^2+2}{2\cdot x^3-x^2}\right)^{x^3}$.

Imamo:

$$\lim_{x\to\infty}\frac{x^3+x^2+2}{2\cdot x^3-x^2}=\left[\begin{array}{l}\text{brojnik i nazivnik}\\ \text{podijelimo potencijom } x^3\end{array}\right]=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{2\cdot x^3}{x^3}-\frac{x^2}{x^3}}=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{2\cdot x^3}{x^3}-\frac{x^2}{x^3}}=$$

$$=\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{2}{x^3}}{2-\frac{1}{x}}=\frac{1+0+0}{2-0}=\frac{1}{2}$$

i

$$\lim_{x\to\infty}x^3=+\infty.$$

Vježba 046

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\frac{x+1}{2\cdot x+1}\right)^{x^2}$.

Rezultat: 0.

Zadatak 047 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to+\infty}\frac{\ln\left(1+e^x\right)}{x}$.

Rješenje 047

Ponovimo!

Ponovimo L' Hospitalovo pravilo [čitaj: Lopitál].

Neka zadana funkcija ima oblik količnika $f(x)=\frac{u(x)}{v(x)}$ i neka je za x = a ona jednaka

$f(a)=\frac{u(a)}{v(a)}=\frac{0}{0}$ ili $f(a)=\frac{u(a)}{v(a)}=\frac{\infty}{\infty}$, tada je prava vrijednost funkcije f(x) za x = a, tj.

$$f(a)=\lim_{x\to a}\frac{u(x)}{v(x)}=\lim_{x\to a}\frac{u'(x)}{v'(x)}=\lim_{x\to a}\frac{u''(x)}{v''(x)}=\lim_{x\to a}\frac{u'''(x)}{v'''(x)}=\ldots\text{ itd,}$$

dok ne dođemo do određene vrijednosti konačne ili beskonačne. Derivira se posebno brojnik i posebno nazivnik, pri čemu se iza svakog deriviranja dobiveni izraz uredi i uvrsti zadani x = a. Zadatak je riješen, čim dobijemo u brojniku ili u nazivniku konačnu ili beskonačnu vrijednost.

Računamo limes:

$$\lim_{x \to +\infty} \frac{\ln\left(1+e^x\right)}{x} = \left[\begin{array}{c} x \to +\infty \\ \text{neodređeni je oblik } \frac{\infty}{\infty}, \text{ L'Hospitalovo pravilo,} \\ \text{deriviramo posebno brojnik, posebno nazivnik} \end{array}\right] = \lim_{x \to +\infty} \frac{\left(\ln\left(1+e^x\right)\right)'}{x'} =$$

$$= \lim_{x \to +\infty} \frac{\frac{1}{1+e^x} \cdot e^x}{1} = \lim_{x \to +\infty} \frac{e^x}{1+e^x} = \left[\begin{array}{c} x \to +\infty \\ \text{neodređeni je oblik } \frac{\infty}{\infty}, \text{ L'Hospitalovo pravilo,} \\ \text{deriviramo posebno brojnik, posebno nazivnik} \end{array}\right] =$$

$$= \lim_{x \to +\infty} \frac{\left(e^x\right)'}{\left(1+e^x\right)'} = \lim_{x \to +\infty} \frac{e^x}{e^x} = \lim_{x \to +\infty} 1 = 1.$$

Vježba 047

Odredite graničnu vrijednost (limes): $\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}}$.

Rezultat: 1.

Zadatak 048 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim_{x \to -\infty} \frac{\ln\left(1+e^x\right)}{x}$.

Rješenje 048

Ponovimo!

Ponovimo L' Hospitalovo pravilo [čitaj: Lopitál].

Neka zadana funkcija ima oblik količnika $f(x) = \frac{u(x)}{v(x)}$ i neka je za x = a ona jednaka

$f(a) = \frac{u(a)}{v(a)} = \frac{0}{0}$ ili $f(a) = \frac{u(a)}{v(a)} = \frac{\infty}{\infty}$, tada je prava vrijednost funkcije f(x) za x = a, tj.

$$f(a) = \lim_{x \to a} \frac{u(x)}{v(x)} = \lim_{x \to a} \frac{u'(x)}{v'(x)} = \lim_{x \to a} \frac{u''(x)}{v''(x)} = \lim_{x \to a} \frac{u'''(x)}{v'''(x)} = \ldots \text{ itd,}$$

dok ne dođemo do određene vrijednosti konačne ili beskonačne. Derivira se posebno brojnik i posebno nazivnik, pri čemu se iza svakog deriviranja dobiveni izraz uredi i uvrsti zadani x = a. Zadatak je riješen, čim dobijemo u brojniku ili u nazivniku konačnu ili beskonačnu vrijednost.

$$a^{-n} = \frac{1}{a^n} \quad , \quad \ln e^n = n \quad , \quad \ln \frac{a}{b} = \ln a - \ln b.$$

Računamo limes

$$\lim_{x \to -\infty} \frac{\ln\left(1+e^x\right)}{x} = \left[\begin{array}{l} \text{zamjena (supstitucija)} \\ x = -t \ \Rightarrow \ t = -x \\ x \to -\infty \ \Rightarrow \ t \to +\infty \end{array}\right] = \lim_{t \to +\infty} \frac{\ln\left(1+e^{-t}\right)}{-t} = -\lim_{t \to +\infty} \frac{\ln\left(1+\frac{1}{e^t}\right)}{t} =$$

$$= -\lim_{t \to +\infty} \frac{\ln \frac{e^t + 1}{e^t}}{t} = -\lim_{t \to +\infty} \frac{\ln\left(e^t + 1\right) - \ln e^t}{t} = -\lim_{t \to +\infty} \frac{\ln\left(1 + e^t\right) - t}{t} =$$

$$= -\lim_{t \to +\infty} \left[\frac{\ln\left(1 + e^t\right)}{t} - \frac{t}{t} \right] = -\lim_{t \to +\infty} \left[\frac{\ln\left(1 + e^t\right)}{t} - 1 \right] = -\lim_{t \to +\infty} \frac{\ln\left(1 + e^t\right)}{t} + 1 =$$

$$= -\left[\begin{array}{l} \lim\limits_{t \to +\infty} \frac{\ln\left(1 + e^t\right)}{t} = 1 \\ \text{Pogledati Zadatak 047} \end{array} \right] + 1 = -1 + 1 = 0.$$

Vježba 048

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to -\infty} \frac{x}{\sqrt{x^2 + 1}}$.

Rezultat: – 1.

Zadatak 049 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 0} \left(x \cdot \sin \frac{\pi}{x} \right)$.

Rješenje 049

Najprije uočimo da $\lim\limits_{x \to 0} \sin \frac{\pi}{x}$ ne postoji, tj. funkcija $y = \sin \frac{\pi}{x}$ u točki x = 0 nema limes (graničnu vrijednost) jer je:

- $\lim\limits_{x \to 0} \sin \frac{\pi}{x} = 0$ za $x = \frac{1}{k}$
- $\lim\limits_{x \to 0} \sin \frac{\pi}{x} = 1$ za $x = \frac{1}{\frac{1}{2} + 2 \cdot k}$
- $\lim\limits_{x \to 0} \sin \frac{\pi}{x} = -1$ za $x = \frac{1}{\frac{3}{2} + 2 \cdot k}$ itd.

Kad se x smanjuje, tj. teži prema nuli preko raznih vrijednosti, dobit će se i razne vrijednosti granične vrijednosti ili limesa. Dakle, $\lim\limits_{x \to 0} \sin \frac{\pi}{x}$ ne postoji.

Zadani limes iznosi:

$$\lim_{x \to 0} \left(x \cdot \sin \frac{\pi}{x} \right) = \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin \frac{\pi}{x} = 0.$$

Dokažimo da vrijedi: $\lim\limits_{x \to 0} \left(x \cdot \sin \frac{\pi}{x} \right) = 0.$

Ponovimo!

Limes funkcije: Kažemo da funkcija f(x) → A (čitaj "ef od x teži prema A") kada x → a (čitaj "x teži prema a") (A i a su brojevi) ili da je

$$\lim_{x \to a} f(x) = A,$$

ako za svaki ε > 0 postoji zakav δ = δ(ε) > 0 da je

$$|f(x)-A|<\varepsilon \text{ za } 0<|x-a|<\delta.$$

Provjeravamo:

$$\lim_{x\to 0}\left(x\cdot\sin\frac{\pi}{x}\right)=0 \Rightarrow \left.\begin{array}{c} f(x)=x\cdot\sin\frac{\pi}{x} \\ A=0 \ , \ a=0\end{array}\right\} \Rightarrow \left[\begin{array}{c}|f(x)-A|<\varepsilon \\ 0<|x-a|<\delta\end{array}\right] \Rightarrow \left|x\cdot\sin\frac{\pi}{x}-0\right|<\varepsilon \Rightarrow$$

$$\Rightarrow \left|x\cdot\sin\frac{\pi}{x}\right|<\varepsilon \Rightarrow |x|\cdot\left|\sin\frac{\pi}{x}\right|<\varepsilon \Rightarrow \left[\begin{array}{c}\text{za funkciju sinus vrijedi} \\ |\sin x|\le 1\end{array}\right] \Rightarrow |x|<\varepsilon \Rightarrow |x-0|<\varepsilon=\delta(\varepsilon).$$

Vježba 049

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\left(x\cdot\sin\frac{1}{x}\right)$.

Rezultat: 0.

Zadatak 050 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\right)$.

Rješenje 050

Ponovimo!

Razlika kvadrata: $(a-b)\cdot(a+b)=a^2-b^2$.

Razlika kubova: $(a-b)\cdot\left(a^2+a\cdot b+b^2\right)=a^3-b^3$.

$$\sqrt[n]{a^n}=a \quad , \quad a^6-b^6=(a-b)\cdot\left(a^5+a^4\cdot b+a^3\cdot b^2+a^2\cdot b^3+a\cdot b^4+b^5\right).$$

1.inačica

$$\lim_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\right)=\left[\begin{array}{l}\text{prvi razlomak proširujemo s } \left(1+\sqrt{x}\right), \\ \text{drugi razlomak proširujemo s } \left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)\end{array}\right]=$$

$$=\lim_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\cdot\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{1+\sqrt[3]{x}+\sqrt[3]{x^2}}\right)=\lim_{x\to 1}\left(\frac{1+\sqrt{x}}{2\cdot\left(1-\sqrt{x^2}\right)}-\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{3\cdot\left(1-\sqrt[3]{x^3}\right)}\right)=$$

$$=\lim_{x\to 1}\left(\frac{1+\sqrt{x}}{2\cdot(1-x)}-\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{3\cdot(1-x)}\right)=\left[\begin{array}{c}\text{razlomke oduzimamo tako da} \\ \text{nađemo zajednički nazivnik}\end{array}\right]=$$

$$=\lim_{x\to 1}\frac{3\cdot\left(1+\sqrt{x}\right)-2\cdot\left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{6\cdot(1-x)}=\lim_{x\to 1}\frac{3+3\cdot\sqrt{x}-2-2\cdot\sqrt[3]{x}-2\cdot\sqrt[3]{x^2}}{6\cdot(1-x)}=$$

$$=\lim_{x\to 1}\frac{1+3\cdot\sqrt{x}-2\cdot\sqrt[3]{x}-2\cdot\sqrt[3]{x^2}}{6\cdot(1-x)}=\left[\begin{array}{l}\text{zamjena (supstitucija)} \\ x=t^6 \Rightarrow x^2=t^{12} \\ x\to 1 \Rightarrow t\to 1\end{array}\right]=\lim_{t\to 1}\frac{1+3\cdot\sqrt{t^6}-2\cdot\sqrt[3]{t^6}-2\cdot\sqrt[3]{t^{12}}}{6\cdot\left(1-t^6\right)}=$$

$$=\lim_{t\to 1}\frac{1+3\cdot t^3-2\cdot t^2-2\cdot t^4}{6\cdot\left(1-t^6\right)}=\left[\begin{array}{l}\qquad\qquad\qquad t\to 1\\ \text{neodređeni je oblik } \frac{0}{0}\text{, L'Hospitalovo pravilo, deriviramo}\\ \text{posebno brojnik, posebno nazivnik po varijabli t}\end{array}\right]=$$

$$=\lim_{t\to 1}\frac{\left(1+3\cdot t^3-2\cdot t^2-2\cdot t^4\right)'}{\left(6\cdot\left(1-t^6\right)\right)'}=\lim_{t\to 1}\frac{0+9\cdot t^2-4\cdot t-8\cdot t^3}{6\cdot\left(-6\cdot t^5\right)}=\lim_{t\to 1}\frac{9\cdot t^2-4\cdot t-8\cdot t^3}{-36\cdot t^5}=$$

$$=\frac{9\cdot 1^2-4\cdot 1-8\cdot 1^3}{-36\cdot 1^5}=\frac{9-4-8}{-36}=\frac{-3}{-36}=\frac{1}{12}.$$

2.inačica

$$\lim_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\right)=\left[\begin{array}{l}\text{prvi razlomak proširujemo s } \left(1+\sqrt{x}\right),\\ \text{drugi razlomak proširujemo s } \left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)\end{array}\right]=$$

$$=\lim_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\cdot\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{1+\sqrt[3]{x}+\sqrt[3]{x^2}}\right)=\lim_{x\to 1}\left(\frac{1+\sqrt{x}}{2\cdot\left(1-\sqrt{x^2}\right)}-\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{3\cdot\left(1-\sqrt[3]{x^3}\right)}\right)=$$

$$=\lim_{x\to 1}\left(\frac{1+\sqrt{x}}{2\cdot(1-x)}-\frac{1+\sqrt[3]{x}+\sqrt[3]{x^2}}{3\cdot(1-x)}\right)=\left[\begin{array}{l}\text{razlomke oduzimamo tako da}\\ \text{nađemo zajednički nazivnik}\end{array}\right]=$$

$$=\lim_{x\to 1}\frac{3\cdot\left(1+\sqrt{x}\right)-2\cdot\left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{6\cdot(1-x)}=\lim_{x\to 1}\frac{3+3\cdot\sqrt{x}-2-2\cdot\sqrt[3]{x}-2\cdot\sqrt[3]{x^2}}{6\cdot(1-x)}=$$

$$=\lim_{x\to 1}\frac{1+3\cdot\sqrt{x}-2\cdot\sqrt[3]{x}-2\cdot\sqrt[3]{x^2}}{6\cdot(1-x)}=\left[\begin{array}{l}\text{zamjena (supstitucija)}\\ x=t^6\ \Rightarrow\ x^2=t^{12}\\ x\to 1\ \Rightarrow\ t\to 1\end{array}\right]=\lim_{t\to 1}\frac{1+3\cdot\sqrt{t^6}-2\cdot\sqrt[3]{t^6}-2\cdot\sqrt[3]{t^{12}}}{6\cdot\left(1-t^6\right)}=$$

$$=\lim_{t\to 1}\frac{1+3\cdot t^3-2\cdot t^2-2\cdot t^4}{6\cdot\left(1-t^6\right)}=\lim_{t\to 1}\frac{-\left(2\cdot t^4-3\cdot t^3+2\cdot t^2-1\right)}{-6\cdot\left(t^6-1\right)}=\lim_{t\to 1}\frac{2\cdot t^4-3\cdot t^3+2\cdot t^2-1}{6\cdot\left(t^6-1\right)}=$$

$$=\left[\begin{array}{l}\text{budući da su brojnik i nazivnik jednaki nuli za t = 1,}\\ \text{moramo brojnik i nazivnik skratiti s binomom t − 1,}\\ \text{stoga brojnik i nazivnik rastavljamo na faktore}\end{array}\right]=\lim_{t\to 1}\frac{2\cdot t^4-3\cdot t^3+2\cdot t^2-1}{6\cdot\left(t^6-1\right)}=$$

$$=\lim_{t\to 1}\frac{2\cdot t^4-2\cdot t^3-t^3+t^2+t^2-1}{6\cdot\left(t^6-1\right)}=\lim_{t\to 1}\frac{2\cdot t^3\cdot(t-1)-t^2\cdot(t-1)+(t-1)\cdot(t+1)}{6\cdot(t-1)\cdot\left(t^5+t^4+t^3+t^2+t+1\right)}=$$

$$=\lim_{t\to 1}\frac{(t-1)\cdot\left(2\cdot t^3-t^2+t+1\right)}{6\cdot(t-1)\cdot\left(t^5+t^4+t^3+t^2+t+1\right)}=\lim_{t\to 1}\frac{2\cdot t^3-t^2+t+1}{6\cdot\left(t^5+t^4+t^3+t^2+t+1\right)}=$$

$$=\frac{2\cdot 1^3-1^2+1+1}{6\cdot\left(1^5+1^4+1^3+1^2+1+1\right)}=\frac{2-1+1+1}{6\cdot(1+1+1+1+1+1)}=\frac{3}{36}=\frac{1}{12}.$$

3.inačica

$$\lim_{x\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{x}\right)}-\frac{1}{3\cdot\left(1-\sqrt[3]{x}\right)}\right)=\begin{bmatrix}\text{zamjena (supstitucija)}\\ x=t^6\\ x\to 1\ \Rightarrow\ t\to 1\end{bmatrix}=\lim_{t\to 1}\left(\frac{1}{2\cdot\left(1-\sqrt{t^6}\right)}-\frac{1}{3\cdot\left(1-\sqrt[3]{t^6}\right)}\right)=$$

$$=\lim_{t\to 1}\left(\frac{1}{2\cdot\left(1-t^3\right)}-\frac{1}{3\cdot\left(1-t^2\right)}\right)=\lim_{t\to 1}\left(\frac{1}{2\cdot(1-t)\cdot\left(1+t+t^2\right)}-\frac{1}{3\cdot(1-t)\cdot(1+t)}\right)=$$

$$=\begin{bmatrix}\text{razlomke oduzimamo tako da}\\ \text{nađemo zajednički nazivnik}\end{bmatrix}=\lim_{t\to 1}\frac{3\cdot(1+t)-2\cdot\left(1+t+t^2\right)}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=$$

$$=\lim_{t\to 1}\frac{3+3\cdot t-2-2\cdot t-2\cdot t^2}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=\lim_{t\to 1}\frac{1+t-2\cdot t^2}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=\begin{bmatrix}\text{brojnik rastavimo na faktore}\\ \text{tako da dobijemo faktor } 1-t\end{bmatrix}=$$

$$=\lim_{t\to 1}\frac{1+t-2\cdot t^2}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=\lim_{t\to 1}\frac{1-t^2+t-t^2}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=\lim_{t\to 1}\frac{(1-t)\cdot(1+t)+t\cdot(1-t)}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=$$

$$=\begin{bmatrix}\text{u brojniku izlučimo faktor}\\ (1-t)\end{bmatrix}=\lim_{t\to 1}\frac{(1-t)\cdot(1+t+t)}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=\lim_{t\to 1}\frac{(1-t)\cdot(1+2\cdot t)}{6\cdot(1-t)\cdot(1+t)\cdot\left(1+t+t^2\right)}=$$

$$=\lim_{t\to 1}\frac{1+2\cdot t}{6\cdot(1+t)\cdot\left(1+t+t^2\right)}=\frac{1+2\cdot 1}{6\cdot(1+1)\cdot\left(1+1+1^2\right)}=\frac{3}{36}=\frac{1}{12}.$$

Vježba 050

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right).$

Rezultat: – 1.

Zadatak 051 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\frac{(2\cdot x+3)^3\cdot(3\cdot x-2)^2}{x^5+5}.$

Rješenje 051

Ponovimo!

Kub zbroja: $(a+b)^3=a^3+3\cdot a^2\cdot b+3\cdot a\cdot b^2+b^3.$ Kvadrat razlike: $(a-b)^2=a^2-2\cdot a\cdot b+b^2.$

$$\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n.$$

Pri traženju limesa količnika dvaju polinoma po varijabli x kada $x\to\infty$, preporučljivo je i brojnik i nazivnik prethodno podijeliti sa x^n, gdje je x^n najviša potencija tih polinoma.

Računamo limes:

1.inačica

$$\lim_{x\to\infty}\frac{(2\cdot x+3)^3\cdot(3\cdot x-2)^2}{x^5+5}=\lim_{x\to\infty}\frac{\left((2\cdot x)^3+3\cdot(2\cdot x)^2\cdot 3+3\cdot 2\cdot x\cdot 3^2+3^3\right)\cdot\left(9\cdot x^2-12\cdot x+4\right)}{x^5+5}=$$

$$=\lim_{x\to\infty}\frac{\left(8\cdot x^3+36\cdot x^2+54\cdot x+27\right)\cdot\left(9\cdot x^2-12\cdot x+4\right)}{x^5+5}=$$

$$=\lim_{x\to\infty}\frac{72x^5-96x^4+32x^3+324x^4-432x^3+144x^2+486x^3-648x^2+216x+243x^2-324x+108}{x^5+5}=$$

$$=\lim_{x\to\infty}\frac{72\cdot x^5+228\cdot x^4+86\cdot x^3-261\cdot x^2-108\cdot x+108}{x^5+5}=\left[\begin{array}{l}\text{dijelimo brojnik i}\\ \text{nazivnik potencijom } x^5\end{array}\right]=$$

$$=\lim_{x\to\infty}\frac{\frac{72\cdot x^5+228\cdot x^4+86\cdot x^3-261\cdot x^2-108\cdot x+108}{x^5}}{\frac{x^5+5}{x^5}}=$$

$$=\lim_{x\to\infty}\frac{\frac{72\cdot x^5}{x^5}+\frac{228\cdot x^4}{x^5}+\frac{86\cdot x^3}{x^5}-\frac{261\cdot x^2}{x^5}-\frac{108\cdot x}{x^5}+\frac{108}{x^5}}{\frac{x^5}{x^5}+\frac{5}{x^5}}=\lim_{x\to\infty}\frac{72+\frac{228}{x}+\frac{86}{x^2}-\frac{261}{x^3}-\frac{108}{x^4}+\frac{108}{x^5}}{1+\frac{5}{x^5}}=$$

$$=\frac{72+0+0-0-0+0}{1+0}=\frac{72}{1}=72.$$

2.inačica

$$\lim_{x\to\infty}\frac{(2\cdot x+3)^3\cdot(3\cdot x-2)^2}{x^5+5}=\left[\begin{array}{l}\text{dijelimo brojnik i}\\ \text{nazivnik potencijom } x^5\end{array}\right]=\lim_{x\to\infty}\frac{\frac{(2\cdot x+3)^3\cdot(3\cdot x-2)^2}{x^5}}{\frac{x^5+5}{x^5}}=$$

$$=\lim_{x\to\infty}\frac{\frac{(2\cdot x+3)^3\cdot(3\cdot x-2)^2}{x^3\cdot x^2}}{\frac{x^5+5}{x^5}}=\lim_{x\to\infty}\frac{\frac{(2\cdot x+3)^3}{x^3}\cdot\frac{(3\cdot x-2)^2}{x^2}}{\frac{x^5+5}{x^5}}=\lim_{x\to\infty}\frac{\left(\frac{2\cdot x+3}{x}\right)^3\cdot\left(\frac{3\cdot x-2}{x}\right)^2}{\frac{x^5+5}{x^5}}=$$

$$=\lim_{x\to\infty}\frac{\left(\frac{2\cdot x}{x}+\frac{3}{x}\right)^3\cdot\left(\frac{3\cdot x}{x}-\frac{2}{x}\right)^2}{\frac{x^5}{x^5}+\frac{5}{x^5}}=\lim_{x\to\infty}\frac{\left(2+\frac{3}{x}\right)^3\cdot\left(3-\frac{2}{x}\right)^2}{1+\frac{5}{x^5}}=\frac{(2+0)^3\cdot(3-0)^2}{1+0}=\frac{2^3\cdot 3^2}{1}=8\cdot 9=72.$$

Vježba 051

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty}\frac{(2\cdot x+3)^2\cdot(3\cdot x-2)^3}{x^5+5}$.

Rezultat: 108.

Zadatak 052 (Sanja, informatika)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\dfrac{1+\cos\pi x}{1-2\cdot x+x^2}$.

Rješenje 052

Ponovimo!

Ponovimo L' Hospitalovo pravilo [čitaj: Lopitál].

Neka zadana funkcija ima oblik količnika $f(x)=\dfrac{u(x)}{v(x)}$ i neka je za x = a ona jednaka

$f(a)=\dfrac{u(a)}{v(a)}=\dfrac{0}{0}$ ili $f(a)=\dfrac{u(a)}{v(a)}=\dfrac{\infty}{\infty}$, tada je prava vrijednost funkcije f(x) za x = a, tj.

$$f(a)=\lim_{x\to a}\frac{u(x)}{v(x)}=\lim_{x\to a}\frac{u'(x)}{v'(x)}=\lim_{x\to a}\frac{u''(x)}{v''(x)}=\lim_{x\to a}\frac{u'''(x)}{v'''(x)}=\ldots \text{ itd,}$$

dok ne dođemo do određene vrijednosti konačne ili beskonačne. Derivira se posebno brojnik i posebno nazivnik, pri čemu se iza svakog deriviranja dobiveni izraz uredi i uvrsti zadani x = a. Zadatak je riješen, čim dobijemo u brojniku ili u nazivniku konačnu ili beskonačnu vrijednost.

Računamo limes:

$$\lim_{x\to 1}\frac{1+\cos\pi x}{1-2\cdot x+x^2}=\begin{bmatrix} x\to 1 \\ \text{neodređeni je oblik } \frac{0}{0}\text{, L'Hospitalovo pravilo,} \\ \text{deriviramo posebno brojnik, posebno nazivnik}\end{bmatrix}=\lim_{x\to 1}\frac{(1+\cos\pi x)'}{\left(1-2\cdot x+x^2\right)'}=$$

$$=\lim_{x\to 1}\frac{1'+(\cos\pi x)'}{1'-(2\cdot x)'+\left(x^2\right)'}=\lim_{x\to 1}\frac{0-\sin\pi x\cdot\pi}{0-2+2\cdot x}=\lim_{x\to 1}\frac{-\pi\cdot\sin\pi x}{-2+2\cdot x}=$$

$$=\begin{bmatrix} x\to 1 \\ \text{neodređeni je oblik } \frac{0}{0}\text{, L'Hospitalovo pravilo,} \\ \text{deriviramo posebno brojnik, posebno nazivnik}\end{bmatrix}=\lim_{x\to 1}\frac{(-\pi\cdot\sin\pi x)'}{(-2+2\cdot x)'}=\lim_{x\to 1}\frac{-\pi\cdot\cos\pi x\cdot\pi}{(-2)'+(2\cdot x)'}=$$

$$=\lim_{x\to 1}\frac{-\pi^2\cdot\cos\pi x}{0+2}=\lim_{x\to 1}\frac{-\pi^2\cdot\cos\pi x}{2}=\frac{-\pi^2\cdot\cos(\pi\cdot 1)}{2}=\frac{-\pi^2\cdot\cos\pi}{2}=\frac{-\pi^2\cdot(-1)}{2}=\frac{\pi^2}{2}.$$

Vježba 052

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\dfrac{\sin\pi x}{1-x}$.

Rezultat: π.

Zadatak 053 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 2}\dfrac{x^2-6\cdot x+8}{x^2-8\cdot x+12}$.

Rješenje 053

Ponovimo!

Kako se trinom rastavlja na faktore pogledati: Algebarski izrazi (1) Zadatak 003 i Zadatak 010.

$$\lim_{x\to 2}\frac{x^2-6\cdot x+8}{x^2-8\cdot x+12}=\lim_{x\to 2}\frac{x^2-6\cdot x+8}{x^2-8\cdot x+12}=\lim_{x\to 2}\frac{x^2-2\cdot x-4\cdot x+8}{x^2-2\cdot x-6\cdot x+12}=\lim_{x\to 2}\frac{x^2-2\cdot x-4\cdot x+8}{x^2-2\cdot x-6\cdot x+12}=$$

$$= \lim_{x \to 2} \frac{x \cdot (x-2) - 4 \cdot (x-2)}{x \cdot (x-2) - 6 \cdot (x-2)} = \lim_{x \to 2} \frac{(x-2) \cdot (x-4)}{(x-2) \cdot (x-6)} = \lim_{x \to 2} \frac{x-4}{x-6} = \frac{2-4}{2-6} = \frac{-2}{-4} = \frac{1}{2}.$$

Vježba 053

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 2} \frac{x^2 + x - 6}{x^2 - x - 2}$.

Rezultat: $\frac{5}{3}$.

Zadatak 054 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{10 \cdot x^2 - 13 \cdot x + 3}{x^2 - 8 \cdot x + 5}$.

Rješenje 054

Ponovimo!

Pri traženju limesa količnika dvaju cijelih polinoma varijable x kada $x \to \infty$, preporučljivo je oba člana količnika prethodno podijeliti s x^p, gdje je p najveći eksponent potencija tih polinoma.

Također je važno zapamtiti sljedeći limes

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

$$\lim_{x \to \infty} \frac{10 \cdot x^2 - 13 \cdot x + 3}{x^2 - 8 \cdot x + 5} = \begin{bmatrix} \text{brojnik i nazivnik} \\ \text{podijelimo potencijom } x^2 \end{bmatrix} = \lim_{x \to \infty} \frac{\frac{10 \cdot x^2 - 13 \cdot x + 3}{x^2}}{\frac{x^2 - 8 \cdot x + 5}{x^2}} =$$

$$= \lim_{x \to \infty} \frac{\frac{10 \cdot x^2}{x^2} - \frac{13 \cdot x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{8 \cdot x}{x^2} + \frac{5}{x^2}} = \lim_{x \to \infty} \frac{10 - \frac{13}{x} + \frac{3}{x^2}}{1 - \frac{8}{x} + \frac{5}{x^2}} = \frac{10 - 0 + 0}{1 - 0 + 0} = 10.$$

Vježba 054

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{10 \cdot x^2 - 13 \cdot x + 9}{x^2 - 8 \cdot x + 9}$.

Rezultat: 10.

Zadatak 055 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{2 \cdot x^2 - 3 \cdot x - 4}{\sqrt{x^4 + 1}}$.

Rješenje 055

Ponovimo!

Pri traženju limesa količnika dvaju cijelih polinoma varijable x kada $x \to \infty$, preporučljivo je oba člana količnika prethodno podijeliti s x^p, gdje je p najveći eksponent potencija tih polinoma.

Također je važno zapamtiti sljedeći limes

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

$$\lim_{x \to \infty} \frac{2 \cdot x^2 - 3 \cdot x - 4}{\sqrt{x^4 + 1}} = \begin{bmatrix} \text{brojnik i nazivnik} \\ \text{podijelimo potencijom } x^2 \end{bmatrix} =$$

$$= \lim_{x\to\infty} \frac{\frac{2\cdot x^2-3\cdot x-4}{x^2}}{\frac{\sqrt{x^4+1}}{x^2}} = \lim_{x\to\infty} \frac{\frac{2\cdot x^2-3\cdot x-4}{x^2}}{\sqrt{\frac{x^4+1}{x^4}}} = \lim_{x\to\infty} \frac{\frac{2\cdot x^2}{x^2}-\frac{3\cdot x}{x^2}-\frac{4}{x^2}}{\sqrt{\frac{x^4}{x^4}+\frac{1}{x^4}}} =$$

$$= \lim_{x\to\infty} \frac{2-\frac{3}{x}-\frac{4}{x^2}}{\sqrt{1+\frac{1}{x^4}}} = \frac{2-0-0}{\sqrt{1+0}} = \frac{2}{1} = 2.$$

Vježba 055

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty} \frac{2\cdot x-4}{\sqrt{x^2+1}}$.

Rezultat: 2.

Zadatak 056 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

Rješenje 056

Ponovimo!

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Razlika kvadrata: $(a-b)\cdot(a+b)=a^2-b^2$.

1.inačica.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \left[\begin{array}{l}\text{brojnik i nazivnik}\\ \text{pomnožimo s } \sqrt{x}+1\end{array}\right] = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{\left(\sqrt{x}\right)^2-1}{(x-1)\cdot\left(\sqrt{x}+1\right)} =$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)\cdot\left(\sqrt{x}+1\right)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

2.inačica.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \left[\begin{array}{c}\text{supstitucija}\\ t^2=x \Rightarrow t=\sqrt{x}\\ x\to 1 \Rightarrow t\to 1\end{array}\right] = \lim_{t\to 1} \frac{\sqrt{t^2}-1}{t^2-1} = \lim_{t\to 1} \frac{t-1}{t^2-1} = \lim_{t\to 1} \frac{t-1}{(t-1)\cdot(t+1)} =$$

$$= \lim_{t\to 1} \frac{1}{t+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Vježba 056

Odredite graničnu vrijednost (limes): $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$.

Rezultat: 2.

Zadatak 057 (Kiki, studentica)

Odredite graničnu vrijednost (limes): $\lim_{x\to -1} \frac{x^2+3\cdot x+2}{x^2+2\cdot x+1}$.

Rješenje 057

Ponovimo!

Kako se trinom rastavlja na faktore pogledati: Algebarski izrazi (1) Zadatak 003 i Zadatak 010.

Kvadrat zbroja: $(a+b)^2=a^2+2\cdot a\cdot b+b^2$.

$$\lim_{x\to -1}\frac{x^2+3\cdot x+2}{x^2+2\cdot x+1}=\lim_{x\to -1}\frac{x^2+3\cdot x+2}{(x+1)^2}=\lim_{x\to -1}\frac{x^2+x+2\cdot x+2}{(x+1)^2}=\lim_{x\to -1}\frac{x^2+x+2\cdot x+2}{(x+1)^2}=$$

$$=\lim_{x\to -1}\frac{x\cdot(x+1)+2\cdot(x+1)}{(x+1)^2}=\lim_{x\to -1}\frac{(x+1)\cdot(x+2)}{(x+1)^2}=\lim_{x\to -1}\frac{(x+1)\cdot(x+2)}{(x+1)^2}=\lim_{x\to -1}\frac{x+2}{x+1}=\frac{-1+2}{-1+1}=\infty.$$

Vježba 057

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\dfrac{x^2+2\cdot x-3}{x^2-2\cdot x+1}$.

Rezultat: ∞.

Zadatak 058 (Marijana, studentica)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{(x-2)^3}$.

Rješenje 058

Ponovimo!

Kako se trinom rastavlja na faktore pogledati: Algebarski izrazi (1) Zadatak 003 i Zadatak 010.

$$\lim_{x\to 2}\frac{x^2-x-2}{(x-2)^3}=\lim_{x\to 2}\frac{x^2-x-2}{(x-2)^3}=\lim_{x\to 2}\frac{x^2+x-2\cdot x-2}{(x-2)^3}=\lim_{x\to 2}\frac{x^2+x-2\cdot x-2}{(x-2)^3}=$$

$$=\lim_{x\to 2}\frac{x\cdot(x+1)-2\cdot(x+1)}{(x-2)^3}=\lim_{x\to 2}\frac{(x+1)\cdot(x-2)}{(x-2)^3}=\lim_{x\to 2}\frac{(x+1)\cdot(x-2)}{(x-2)^3}=\lim_{x\to 2}\frac{x+1}{(x-2)^2}=\frac{2+1}{(-2+2)^2}=\infty.$$

Vježba 058

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{(x-2)^2}$.

Rezultat: ∞.

Zadatak 059 (Tanja, Kiki, studentice)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1}\dfrac{2^{2\cdot x+1}-3\cdot 2^x-2}{\sqrt{6-2^x}-2}$.

Rješenje 059

Ponovimo!

$$a^n\cdot a^m=a^{n+m}\quad,\quad \left(a^n\right)^m=a^{n\cdot m}\quad,\quad (a-b)\cdot(a+b)=a^2-b^2.$$

$$\lim_{x\to 1}\frac{2^{2\cdot x+1}-3\cdot 2^x-2}{\sqrt{6-2^x}-2}=\left[\begin{array}{l}\text{racionalizacija}\\ \text{nazivnika}\end{array}\right]=\lim_{x\to 1}\frac{2^{2\cdot x+1}-3\cdot 2^x-2}{\sqrt{6-2^x}-2}\cdot\frac{\sqrt{6-2^x}+2}{\sqrt{6-2^x}+2}=$$

$$=\lim_{x\to 1}\frac{\left(2^{2\cdot x+1}-3\cdot 2^x-2\right)\cdot\left(\sqrt{6-2^x}+2\right)}{\left(\sqrt{6-2^x}\right)^2-2^2}=\lim_{x\to 1}\frac{\left(2^{2\cdot x}\cdot 2^1-3\cdot 2^x-2\right)\cdot\left(\sqrt{6-2^x}+2\right)}{6-2^x-4}=$$

$$= \lim_{x\to 1} \frac{\left(2^{2\cdot x}\cdot 2-3\cdot 2^{x}-2\right)\cdot\left(\sqrt{6-2^{x}}+2\right)}{2-2^{x}} = \lim_{x\to 1} \frac{\left(2\cdot\left(2^{x}\right)^{2}-4\cdot 2^{x}+2^{x}-2\right)\cdot\left(\sqrt{6-2^{x}}+2\right)}{2-2^{x}} =$$

$$= \lim_{x\to 1} \frac{\left(2\cdot 2^{x}\cdot\left(2^{x}-2\right)+\left(2^{x}-2\right)\right)\cdot\left(\sqrt{6-2^{x}}+2\right)}{-\left(2^{x}-2\right)} = \lim_{x\to 1} \frac{\left(2^{x}-2\right)\cdot\left(2\cdot 2^{x}+1\right)\cdot\left(\sqrt{6-2^{x}}+2\right)}{-\left(2^{x}-2\right)} =$$

$$= \lim_{x\to 1} \frac{\left(2\cdot 2^{x}+1\right)\cdot\left(\sqrt{6-2^{x}}+2\right)}{-1} = -\lim_{x\to 1}\left(2^{x+1}+1\right)\cdot\left(\sqrt{6-2^{x}}+2\right) = -\left(2^{1+1}+1\right)\cdot\left(\sqrt{6-2^{1}}+2\right) =$$

$$= -\left(2^{2}+1\right)\cdot\left(\sqrt{6-2}+2\right) = -(4+1)\cdot\left(\sqrt{4}+2\right) = -5\cdot(2+2) = -5\cdot 4 = -20.$$

Vježba 059

Odredite graničnu vrijednost (limes): $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}.$

Rezultat: $\frac{1}{2\cdot\sqrt{a}}.$

Zadatak 060 (Marina, studentica)

Odredite graničnu vrijednost (limes): $\lim_{x\to 1}\left(\frac{3}{1-x^{3}}-\frac{1}{1-x}\right).$

Rješenje 060

Ponovimo!

$$a^{2}-b^{2}=(a-b)\cdot(a+b), \quad a^{3}-b^{3}=(a-b)\cdot\left(a^{2}+a\cdot b+b^{2}\right).$$

$$\lim_{x\to 1}\left(\frac{3}{1-x^{3}}-\frac{1}{1-x}\right) = \lim_{x\to 1}\left(\frac{3}{(1-x)\cdot\left(1+x+x^{2}\right)}-\frac{1}{1-x}\right) = \lim_{x\to 1}\frac{3-\left(1+x+x^{2}\right)}{(1-x)\cdot\left(1+x+x^{2}\right)} =$$

$$= \lim_{x\to 1}\frac{3-1-x-x^{2}}{(1-x)\cdot\left(1+x+x^{2}\right)} = \lim_{x\to 1}\frac{2-x-x^{2}}{(1-x)\cdot\left(1+x+x^{2}\right)} = \lim_{x\to 1}\frac{1-x+1-x^{2}}{(1-x)\cdot\left(1+x+x^{2}\right)} =$$

$$= \lim_{x\to 1}\frac{(1-x)+\left(1-x^{2}\right)}{(1-x)\cdot\left(1+x+x^{2}\right)} = \lim_{x\to 1}\frac{(1-x)+(1-x)\cdot(1+x)}{(1-x)\cdot\left(1+x+x^{2}\right)} = \left[\begin{array}{c}\text{u brojniku izlučimo}\\ 1-x\end{array}\right] =$$

$$= \lim_{x\to 1}\frac{(1-x)\cdot(1+1+x)}{(1-x)\cdot\left(1+x+x^{2}\right)} = \lim_{x\to 1}\frac{(1-x)\cdot(2+x)}{(1-x)\cdot\left(1+x+x^{2}\right)} = \lim_{x\to 1}\frac{2+x}{1+x+x^{2}} = \frac{2+1}{1+1+1^{2}} = \frac{3}{3} = 1.$$

Vježba 060

Odredite graničnu vrijednost (limes): $\lim_{x\to 1}\frac{x^{3}-1}{x-1}.$

Rezultat: 3.