Zadatak 041 (Josip, gimnazija)

Zadani su vektori $\vec{a} = 3\cdot\vec{i} - \vec{j} - 2\cdot\vec{k}$ i $\vec{b} = \vec{i} + 2\cdot\vec{j} - \vec{k}$. Izračunajte $\left(2\cdot\vec{a} + \vec{b}\right)\times\vec{b}$.

Rješenje 041

Ponovimo!

Vektorski produkt dvaju vektora je vektor i definira se samo za vektore u trodimenzionalnom prostoru.

Ako su zadani vektori $\vec{a} = a_x\cdot\vec{i} + a_y\cdot\vec{j} + a_z\cdot\vec{k}$ i $\vec{b} = b_x\cdot\vec{i} + b_y\cdot\vec{j} + b_z\cdot\vec{k}$, njihov vektorski produkt glasi:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \vec{i}\cdot\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \vec{j}\cdot\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \vec{k}\cdot\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} =$$

$$= \vec{i}\cdot\left(a_y\cdot b_z - a_z\cdot b_y\right) - \vec{j}\cdot\left(a_x\cdot b_z - a_z\cdot b_x\right) + \vec{k}\cdot\left(a_x\cdot b_y - a_y\cdot b_x\right) =$$

$$= \left(a_y\cdot b_z - a_z\cdot b_y\right)\cdot\vec{i} + \left(a_z\cdot b_x - a_x\cdot b_z\right)\cdot\vec{j} + \left(a_x\cdot b_y - a_y\cdot b_x\right)\cdot\vec{k}.$$

$$\vec{a}\times\vec{a} = \vec{0}.$$

$$\left(2\cdot\vec{a} + \vec{b}\right)\times\vec{b} = 2\cdot\vec{a}\times\vec{b} + \vec{b}\times\vec{b} = 2\cdot\vec{a}\times\vec{b} + \vec{0} = 2\cdot\vec{a}\times\vec{b} = 2\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} =$$

$$= 2\cdot\left[\vec{i}\cdot\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} - \vec{j}\cdot\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} + \vec{k}\cdot\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}\right] =$$

$$= 2\cdot\left[\vec{i}\cdot\left((-1)\cdot(-1) - (-2)\cdot 2\right) - \vec{j}\cdot\left(3\cdot(-1) - (-2)\cdot 1\right) + \vec{k}\cdot\left(3\cdot 2 - (-1)\cdot 1\right)\right] =$$

$$= 2\cdot\left[\vec{i}\cdot(1+4) - \vec{j}\cdot(-3+2) + \vec{k}\cdot(6+1)\right] = 2\cdot\left[5\cdot\vec{i} + \vec{j} + 7\cdot\vec{k}\right] = 10\cdot\vec{i} + 2\cdot\vec{j} + 14\cdot\vec{k}.$$

Vježba 041

Zadani su vektori $\vec{a} = 3\cdot\vec{i} - \vec{j} - 2\cdot\vec{k}$ i $\vec{b} = \vec{i} + 2\cdot\vec{j} - \vec{k}$. Izračunajte $\left(3\cdot\vec{a} + \vec{b}\right)\times\vec{b}$.

Rezultat: $15\cdot\vec{i} + 3\cdot\vec{j} + 21\cdot\vec{k}$.

Zadatak 042 (Gaby, maturantica)

Pojednostavnite izraz: $\left(2\cdot\vec{a} + \vec{b}\right)\times\left(\vec{c} - \vec{a}\right) + \left(\vec{b} + \vec{c}\right)\times\left(\vec{a} + \vec{b}\right)$.

Rješenje 042

Ponovimo!

$$\left(\vec{a} + \vec{b}\right)\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c} \quad , \quad \vec{a}\times\vec{a} = \vec{0}$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \quad , \quad \left(\lambda\cdot\vec{a}\right)\times\vec{b} = \lambda\cdot\left(\vec{a}\times\vec{b}\right).$$

$$\left(2\cdot\vec{a} + \vec{b}\right)\times\left(\vec{c} - \vec{a}\right) + \left(\vec{b} + \vec{c}\right)\times\left(\vec{a} + \vec{b}\right) =$$

$$2\cdot\vec a\times\vec c-2\cdot\vec a\times\vec a+\vec b\times\vec c-\vec b\times\vec a+\vec b\times\vec a+\vec b\times\vec b+\vec c\times\vec a+\vec c\times\vec b=$$

$$=2\cdot\vec a\times\vec c-2\cdot\vec a\times\vec a+\vec b\times\vec c-\vec b\times\vec a+\vec b\times\vec a+\vec b\times\vec b+\vec c\times\vec a+\vec c\times\vec b=$$

$$=2\cdot\vec a\times\vec c-2\cdot\underbrace{\vec a\times\vec a}_{\vec 0}+\vec b\times\vec c+\underbrace{\vec b\times\vec b}_{\vec 0}+\vec c\times\vec a+\vec c\times\vec b=$$

$$=2\cdot\vec a\times\vec c+\vec b\times\vec c+\vec c\times\vec a+\vec c\times\vec b=2\cdot\vec a\times\vec c+\vec b\times\vec c-\vec a\times\vec c-\vec b\times\vec c=$$

$$=2\cdot\vec a\times\vec c+\vec b\times\vec c-\vec a\times\vec c-\vec b\times\vec c=2\cdot\vec a\times\vec c-\vec a\times\vec c=\vec a\times\vec c.$$

Vježba 042

Pojednostavnite izraz: $\left(\vec a+\vec b\right)\times\left(\vec a-\vec b\right)$.

Rezultat: $2\cdot\vec b\times\vec a.$

Zadatak 043 (Ante, maturant)

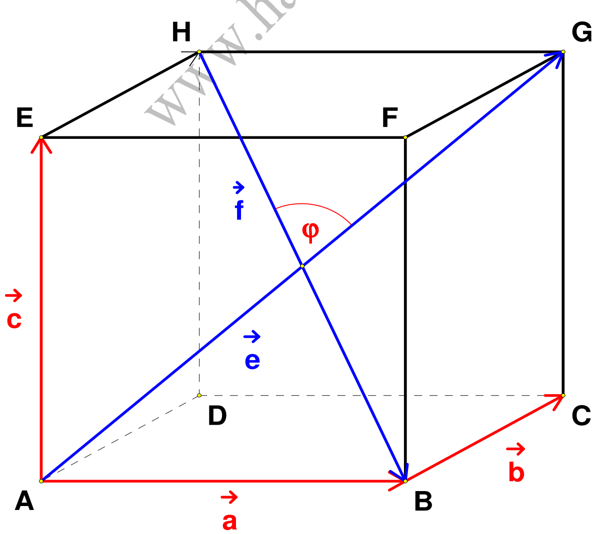
Nađi kut između prostornih dijagonala kocke.

Rješenje 043

Ponovimo!

$$\vec x\circ\vec x=\left|\vec x\right|^2=x^2\quad,\quad x=\sqrt{\vec x\circ\vec x}\quad,\quad \vec x\perp\vec y\;\Rightarrow\;\vec x\circ\vec y=0.$$

$$\vec x\circ\vec y=\left|\vec x\right|\cdot\left|\vec y\right|\cdot\cos\varphi\;,\;\varphi=\angle\left(\vec x,\vec y\right).$$



Sa slike vidi se:

$$\overrightarrow{AB}=\vec a\;,\;\overrightarrow{BC}=\vec b\;,\;\overrightarrow{AE}=\vec c\;,\;\overrightarrow{CG}=\overrightarrow{AE}=\vec c\;,\;\overrightarrow{HD}=-\overrightarrow{AE}=-\vec c\;,\;\overrightarrow{DA}=-\overrightarrow{BC}=-\vec b.$$

$$\left|\vec a\right|=\left|\vec b\right|=\left|\vec c\right|=a\;,\;\vec a\perp\vec b\;\Rightarrow\;\vec a\circ\vec b=0\;,\;\vec a\perp\vec c\;\Rightarrow\;\vec a\circ\vec c=0\;,\;\vec b\perp\vec c\;\Rightarrow\;\vec b\circ\vec c=0.$$

Prostorne dijagonale kocke:

- $\vec{e} = \left|\vec{AG}\right| = \left|\vec{AB}\right| + \left|\vec{BC}\right| + \left|\vec{CG}\right| = \left|\vec{AB}\right| + \left|\vec{BC}\right| + \left|\vec{AE}\right| = \vec{a} + \vec{b} + \vec{c}$
- $\vec{f} = \left|\vec{HB}\right| = \left|\vec{HD}\right| + \left|\vec{DA}\right| + \left|\vec{AB}\right| = -\left|\vec{AE}\right| - \left|\vec{BC}\right| + \left|\vec{AB}\right| = -\vec{c} - \vec{b} + \vec{a} = \vec{a} + \vec{b} + \vec{c}$.

Računamo skalarni produkt:

$$\vec{e} \circ \vec{f} = \left(\vec{a} + \vec{b} + \vec{c}\right) \circ \left(\vec{a} - \vec{b} - \vec{c}\right) =$$

$$= \vec{a} \circ \vec{a} - \vec{a} \circ \vec{b} - \vec{a} \circ \vec{c} + \vec{b} \circ \vec{a} - \vec{b} \circ \vec{b} - \vec{b} \circ \vec{c} + \vec{c} \circ \vec{a} - \vec{c} \circ \vec{b} - \vec{c} \circ \vec{c} = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 - \left|\vec{c}\right|^2 =$$

$$= a^2 - b^2 - c^2 = a^2 - a^2 - a^2 = a^2 - a^2 - a^2 = -a^2.$$

Duljine vektora prostornih dijagonala:

- $$\left|\vec{e}\right| = \sqrt{\vec{e} \circ \vec{e}} = \sqrt{\left(\vec{a} + \vec{b} + \vec{c}\right) \circ \left(\vec{a} + \vec{b} + \vec{c}\right)} =$$

$$= \sqrt{\vec{a} \circ \vec{a} + \vec{a} \circ \vec{b} + \vec{a} \circ \vec{c} + \vec{b} \circ \vec{a} + \vec{b} \circ \vec{b} + \vec{b} \circ \vec{c} + \vec{c} \circ \vec{a} + \vec{c} \circ \vec{b} + \vec{c} \circ \vec{c}} = \sqrt{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2} =$$

$$= \sqrt{a^2 + b^2 + c^2} = \sqrt{a^2 + a^2 + a^2} = \sqrt{3 \cdot a^2} = a \cdot \sqrt{3}.$$

- $$\left|\vec{f}\right| = \sqrt{\vec{f} \circ \vec{f}} = \sqrt{\left(\vec{a} - \vec{b} - \vec{c}\right) \circ \left(\vec{a} - \vec{b} - \vec{c}\right)} =$$

$$= \sqrt{\vec{a} \circ \vec{a} - \vec{a} \circ \vec{b} - \vec{a} \circ \vec{c} - \vec{b} \circ \vec{a} + \vec{b} \circ \vec{b} + \vec{b} \circ \vec{c} - \vec{c} \circ \vec{a} + \vec{c} \circ \vec{b} + \vec{c} \circ \vec{c}} = \sqrt{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2} =$$

$$= \sqrt{a^2 + b^2 + c^2} = \sqrt{a^2 + a^2 + a^2} = \sqrt{3 \cdot a^2} = a \cdot \sqrt{3}.$$

Kut između prostornih dijagonala kocke iznosi:

$$\vec{e} \circ \vec{f} = \left|\vec{e}\right| \cdot \left|\vec{f}\right| \cdot \cos\varphi \Rightarrow \cos\varphi = \frac{\vec{e} \circ \vec{f}}{\left|\vec{e}\right| \cdot \left|\vec{f}\right|} \Rightarrow \cos\varphi = -\frac{a^2}{a \cdot \sqrt{3} \cdot a \cdot \sqrt{3}} \Rightarrow \cos\varphi = -\frac{a^2}{3 \cdot a^2} \Rightarrow$$

$$\Rightarrow \cos\varphi = -\frac{a^2}{3 \cdot a^2} \Rightarrow \cos\varphi = -\frac{1}{3} \Rightarrow \varphi = \cos^{-1}\left(-\frac{1}{3}\right) \Rightarrow \varphi = 109^0\ 28'\ 16''.$$

Vježba 043

Nađi kut između dijagonala kvadrata.

Rezultat: 90º.

Zadatak 044 (Nina, studentica)

Ako sile $\vec{F_1}$ i $\vec{F_2}$ zatvaraju kut 0, i ako je $\left|\vec{F_1}\right| = 15\ N$, $\left|\vec{F_2}\right| = 20\ N$, onda je duljina njihove rezultante jednaka?

Rješenje 044

Ponovimo!

Ako vektori $\vec{a}$ i $\vec{b}$, čije su duljine $\left|\vec{a}\right|$ i $\left|\vec{b}\right|$, zatvaraju međusobno kut od 0 radijana ili 0° onda duljina njihove rezultante $\vec{c}$ glasi

$$\left|\vec{c}\right| = \left|\vec{a}\right| + \left|\vec{b}\right|.$$

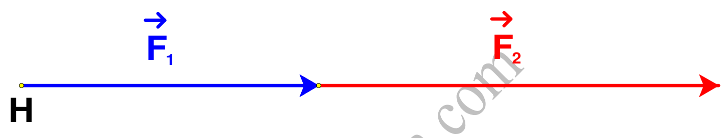
Budući da sile $\vec{F_1}$ i $\vec{F_2}$ zatvaraju kut 0, onda je duljina njihove rezultante $\vec{F}$ jednaka

$$\left.\begin{array}{l} \left|\vec{F_1}\right| = 15\ N\ ,\ \left|\vec{F_2}\right| = 20\ N \\ \left|\vec{F}\right| = \left|\vec{F_1}\right| + \left|\vec{F_2}\right| \end{array}\right\} \Rightarrow \left|\vec{F}\right| = 15\ N + 20\ N = 35\ N.$$

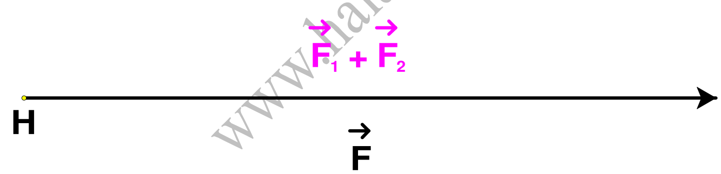
Gledaj slike!

Vektori $\vec{F_1}$ i $\vec{F_2}$ zatvaraju kut od 0 radijana. Hvatište vektora je točka H.

Na kraj vektora $\vec{F_1}$ "nadoveže se" hvatište (početak) vektora $\vec{F_2}$.



Rezultanta vektora $\vec{F_1}$ i $\vec{F_2}$ je vektor $\vec{F}$.



Vježba 044

Ako sile $\vec{F_1}$ i $\vec{F_2}$ zatvaraju kut 0, i ako je $\left|\vec{F_1}\right| = 25\ N, \left|\vec{F_2}\right| = 30\ N,$ onda je duljina njihove rezultante jednaka?

Rezultat: 55 N.

Zadatak 045 (Lucija, gimnazija)

Zadana su dva vektora $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ i $\vec{b} = \vec{i} - 5 \cdot \vec{j} + 2 \cdot \vec{k}$. Koliki je kut među njima?

Rješenje 045

Ponovimo!

Ako su $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k}$ i $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j} + b_z \cdot \vec{k}$ dva vektora, tada vrijedi:

$$\vec{a} \circ \vec{b} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z \quad , \quad \left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad , \quad \left|\vec{b}\right| = \sqrt{b_x^2 + b_y^2 + b_z^2}.$$

$$\cos\alpha = \frac{\vec{a} \circ \vec{b}}{\left|\vec{a}\right| \cdot \left|\vec{b}\right|} \Rightarrow \cos\alpha = \frac{a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}.$$

Izračunamo najprije skalarni produkt vektora $\vec{a}$ i $\vec{b}$, te njihove duljine $\left|\vec{a}\right|$ i $\left|\vec{b}\right|$.

$$\left.\begin{array}{l}\vec{a}=\vec{i}+\vec{j}+\vec{k}\\ \vec{b}=\vec{i}-5\cdot\vec{j}+2\cdot\vec{k}\\ \vec{a}\circ\vec{b}=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z\end{array}\right\}\Rightarrow\ \vec{a}\circ\vec{b}=1\cdot1+1\cdot(-5)+1\cdot2\ \Rightarrow\ \vec{a}\circ\vec{b}=1-5+2\ \Rightarrow\ \vec{a}\circ\vec{b}=-2.$$

$$\left.\begin{array}{l}\vec{a}=\vec{i}+\vec{j}+\vec{k}\\ \left|\vec{a}\right|=\sqrt{a_x^2+a_y^2+a_z^2}\end{array}\right\}\Rightarrow\ \left|\vec{a}\right|=\sqrt{1^2+1^2+1^2}\ \Rightarrow\ \left|\vec{a}\right|=\sqrt{1+1+1}\ \Rightarrow\ \left|\vec{a}\right|=\sqrt{3}.$$

$$\left.\begin{array}{l}\vec{b}=\vec{i}-5\cdot\vec{j}+2\cdot\vec{k}\\ \left|\vec{b}\right|=\sqrt{b_x^2+b_y^2+b_z^2}\end{array}\right\}\Rightarrow\ \left|\vec{b}\right|=\sqrt{1^2+(-5)^2+2^2}\ \Rightarrow\ \left|\vec{b}\right|=\sqrt{1+25+4}\ \Rightarrow\ \left|\vec{b}\right|=\sqrt{30}.$$

Kut među vektorima iznosi:

$$\left.\begin{array}{l}\vec{a}\circ\vec{b}=-2\ ,\ \left|\vec{a}\right|=\sqrt{3}\ ,\ \left|\vec{b}\right|=\sqrt{30}\\ \cos\alpha=\dfrac{\vec{a}\circ\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}\end{array}\right\}\Rightarrow\ \cos\alpha=\frac{-2}{\sqrt{3}\cdot\sqrt{30}}\ \Rightarrow\ \cos\alpha=\frac{-2}{\sqrt{90}}\ \Rightarrow$$

$$\Rightarrow\ \alpha=\cos^{-1}\left(\frac{-2}{\sqrt{90}}\right)\Rightarrow\ \alpha=102.1703233643^{0}.$$

Preračunavamo u stupnjeve, minute i sekunde (ako nemamo računalo koje se može nabaviti u svakoj boljoj trgovini ☺):

$$\alpha=102.1703233643^{0}\ \Rightarrow\ \alpha=102^{0}+0.1703233643^{0}.$$

$$\alpha_1=0.1703233643\cdot60'=10.219401858'$$

$$\alpha_1=10.219401858'\ \Rightarrow\ \alpha_1=10'+0.219401858'.$$

$$\alpha_2=0.219401858\cdot60''=13.16411148''.$$

$$\alpha_2=13.16411148''\ \Rightarrow\ \alpha_2=13''+0.16411148''.$$

Dakle, kut među vektorima ima vrijednost:

$$\alpha=102^{0}\,10'\,13''.$$

Vježba 045

Zadana su dva vektora $\vec{a}=\vec{i}+2\cdot\vec{j}+3\cdot\vec{k}$ i $\vec{b}=\vec{i}-\vec{j}-\vec{k}$. Koliki je kut među njima?

Rezultat: $\alpha=128^{0}\,6'\,47''.$

Zadatak 046 (Lucija, gimnazija)

Zadana su tri vektora: $\vec{a}=3\cdot\vec{i}+2\cdot\vec{j}+\vec{k}$, $\vec{b}=\vec{i}-5\cdot\vec{j}+2\cdot\vec{k}$, $\vec{c}=3\cdot\vec{i}+2\cdot\vec{j}+\vec{k}$.

Nađi $\vec{a}\circ\left(\vec{b}\times\vec{c}\right)$.

Rješenje 046

Ponovimo!

Ako su $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k}$ i $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j} + b_z \cdot \vec{k}$ dva vektora, tada vrijedi:

$$\vec{a} \circ \vec{b} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z.$$

Formula za vektorski produkt vektora $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k}$ i $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j} + b_z \cdot \vec{k}$ pomoću njihovih komponenata u pravokutnom koordinatnom sustavu (Kartezijevom koordinatnom sustavu) glasi:

$$\vec{a} \times \vec{b} = \vec{i} \cdot \left(a_y \cdot b_z - a_z \cdot b_y\right) + \vec{j} \cdot \left(a_z \cdot b_x - a_x \cdot b_z\right) + \vec{k} \cdot \left(a_x \cdot b_y - a_y \cdot b_x\right).$$

Najprije izračunamo vektorski produkt u zagradi:

$$\left.\begin{array}{l} \vec{b} = \vec{i} - 5 \cdot \vec{j} + 2 \cdot \vec{k} \\ \vec{c} = 3 \cdot \vec{i} + 2 \cdot \vec{j} + \vec{k} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \vec{b} \times \vec{c} = \vec{i} \cdot \left(b_y \cdot c_z - b_z \cdot c_y\right) + \vec{j} \cdot \left(b_z \cdot c_x - b_x \cdot c_z\right) + \vec{k} \cdot \left(b_x \cdot c_y - b_y \cdot c_x\right) =$$

$$= \vec{i} \cdot (-5 \cdot 1 - 2 \cdot 2) + \vec{j} \cdot (2 \cdot 3 - 1 \cdot 1) + \vec{k} \cdot (1 \cdot 2 - (-5) \cdot 3) = \vec{i} \cdot (-5 - 4) + \vec{j} \cdot (6 - 1) + \vec{k} \cdot (2 + 15) =$$

$$= -9 \cdot \vec{i} + 5 \cdot \vec{j} + 17 \cdot \vec{k}.$$

Sada računamo $\vec{a} \circ \left(\vec{b} \times \vec{c}\right)$.

$$\left.\begin{array}{l} \vec{a} = 3 \cdot \vec{i} + 2 \cdot \vec{j} + \vec{k} \\ \vec{b} \times \vec{c} = -9 \cdot \vec{i} + 5 \cdot \vec{j} + 17 \cdot \vec{k} \\ \vec{a} \circ \left(\vec{b} \times \vec{c}\right) \end{array}\right\} \Rightarrow \vec{a} \circ \left(\vec{b} \times \vec{c}\right) = \left(3 \cdot \vec{i} + 2 \cdot \vec{j} + \vec{k}\right) \circ \left(-9 \cdot \vec{i} + 5 \cdot \vec{j} + 17 \cdot \vec{k}\right) \Rightarrow$$

$$\Rightarrow \vec{a} \circ \left(\vec{b} \times \vec{c}\right) = 3 \cdot (-9) + 2 \cdot 5 + 1 \cdot 17 \Rightarrow \vec{a} \circ \left(\vec{b} \times \vec{c}\right) = -27 + 10 + 17 \Rightarrow \vec{a} \circ \left(\vec{b} \times \vec{c}\right) = 0.$$

Vježba 046

Zadana su tri vektora: $\vec{a} = 3 \cdot \vec{i} + 2 \cdot \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 5 \cdot \vec{j} + 2 \cdot \vec{k}$, $\vec{c} = 3 \cdot \vec{i} + 2 \cdot \vec{j} + \vec{k}$.

Nađi $\left(\vec{b} \times \vec{c}\right) \circ \vec{a}$.

Rezultat: 0.

Zadatak 047 (Kata, srednja škola)

Nađi kut među vektorima: $\overrightarrow{AB} = -3 \cdot \vec{i} - 4 \cdot \vec{j}$ i $\overrightarrow{CD} = 3 \cdot \vec{i} - 4 \cdot \vec{j}$.

Rješenje 047

Ponovimo!

Ako su $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j}$ i $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j}$ dva vektora, tada za kut α među njima vrijedi:

$$\cos \alpha = \frac{a_x \cdot b_x + a_y \cdot b_y}{\sqrt{a_x^2 + a_y^2} \cdot \sqrt{b_x^2 + b_y^2}}.$$

Kut među zadanim vektorima iznosi:

$$\left.\begin{aligned}&\overrightarrow{AB}=-3\cdot\vec{i}-4\cdot\vec{j}\ ,\ \overrightarrow{CD}=3\cdot\vec{i}-4\cdot\vec{j}\\&\cos\alpha=\frac{a_x\cdot b_x+a_y\cdot b_y}{\sqrt{a_x^2+a_y^2}\cdot\sqrt{b_x^2+b_y^2}}\end{aligned}\right\}\Rightarrow\cos\alpha=\frac{-3\cdot3+(-4)\cdot(-4)}{\sqrt{(-3)^2+(-4)^2}\cdot\sqrt{3^2+(-4)^2}}\Rightarrow$$

$$\Rightarrow\cos\alpha=\frac{-9+16}{\sqrt{9+16}\cdot\sqrt{9+16}}\Rightarrow\cos\alpha=\frac{7}{\sqrt{25}\cdot\sqrt{25}}\Rightarrow\cos\alpha=\frac{7}{5\cdot5}\Rightarrow\cos\alpha=\frac{7}{25}\Rightarrow$$

$$\Rightarrow\alpha=\cos^{-1}\left(\frac{7}{25}\right)\Rightarrow\alpha=73^0\,44'\,23''.$$

Vježba 047

Nađi kut među vektorima: $\overrightarrow{AB}=3\cdot\vec{i}+4\cdot\vec{j}$ i $\overrightarrow{CD}=-3\cdot\vec{i}+4\cdot\vec{j}$.

Rezultat: $\alpha=73^0\,44'\,23''$.

Zadatak 048 (Rockvampirica, matematička gimnazija)

Dokaži da su dijagonale romba međusobno okomite.

Rješenje 048

Ponovimo!

Četverokut s okomitim dijagonalama koji ima barem jednu os simetrije zove se deltoid. Romb je deltoid kojemu sjecište dijagonala raspolavlja dijagonale.

- Stranice romba su sukladne.
- Nasuprotni kutovi romba su sukladni.
- Kutovi uza svaku stranicu romba su suplementni (zbroj im je 180°).
- Romb ima dvije osi simetrije.

Skalarni produkt vektora $\vec{a}$ i $\vec{b}$ je skalar (broj) koji označavamo sa $\vec{a}\circ\vec{b}$ i definiramo ovako:

$$\vec{a}\circ\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi,$$

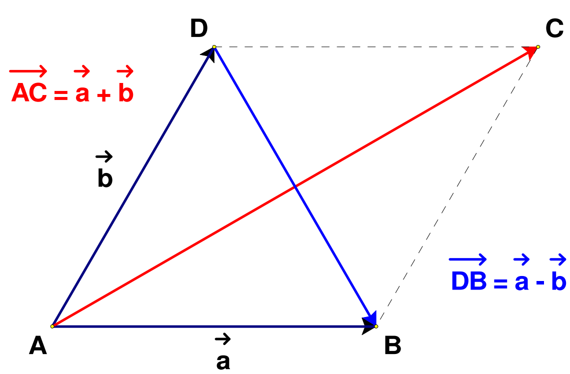
gdje je φ kut između vektora $\vec{a}$ i $\vec{b}$ i uzimamo da je $0\le\varphi\le\pi$.

Vrijedi:

$$\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos0\Rightarrow\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot1\Rightarrow\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\Rightarrow\vec{a}\circ\vec{a}=\left|\vec{a}\right|^2.$$

Dva su vektora međusobno okomita ako je njihov skalarni produkt jednak nuli.

$$\vec{a}\perp\vec{b}\Rightarrow\vec{a}\circ\vec{b}=0.$$



Sa slike vidi se da je:

$$\vec{AC} = \vec{a} + \vec{b} \quad , \quad \vec{DB} = \vec{a} - \vec{b}.$$

Budući da su kod romba sve stranice sukladne, slijedi:

$$\left|\vec{a}\right| = \left|\vec{b}\right| \Rightarrow \left|\vec{a}\right|^2 = \left|\vec{b}\right|^2.$$

Dijagonale romba su okomite ako je njihov skalarni produkt jednak nuli, tj. ako vrijedi:

$$\vec{AC} \circ \vec{DB} = 0.$$

Provjeravamo:

$$\vec{AC} \circ \vec{DB} = \left(\vec{a} + \vec{b}\right) \circ \left(\vec{a} - \vec{b}\right) = \vec{a} \circ \vec{a} - \vec{a} \circ \vec{b} + \vec{b} \circ \vec{a} - \vec{b} \circ \vec{b} =$$

$$= \vec{a} \circ \vec{a} - \vec{a} \circ \vec{b} + \vec{b} \circ \vec{a} - \vec{b} \circ \vec{b} = \vec{a} \circ \vec{a} - \vec{b} \circ \vec{b} = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 =$$

$$= \left[\left|\vec{a}\right| = \left|\vec{b}\right|\right] = 0.$$

Vježba 048

Dokaži da su dijagonale kvadrata međusobno okomite.

Rezultat: Dokaz analogan.

Zadatak 049 (Maja, gimnazija)

U koordinatnom sustavu zadane su točke A(5, 1) i B(6, 3).

a) Prikaži vektor $\vec{AB}$ kao linearnu kombinaciju jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$ te odredi duljine vektora $\vec{OA}$ i $\vec{AB}$.

b) Odredi točku C tako da je $\vec{OC} = \vec{AB}$.

c) Odredi mjeru kuta $\angle AOC$. (Zaokruži rezultat na najbliži cijeli stupanj.)

Rješenje 049

Ponovimo!

Ako su dane točke A(x_1, y_1) i B(x_2, y_2), onda su koordinate vektora koji ih spaja:

$$\vec{AB} = \left(x_2 - x_1\right) \cdot \vec{i} + \left(y_2 - y_1\right) \cdot \vec{j}.$$

Ako su $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j}$, $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j}$ dva vektora, oni su jednaki ako i samo ako su im odgovarajuće koordinate jednake, tj. $a_x = b_x$ i $a_y = b_y$.

Ako su $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j}$, $\vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j}$ dva vektora, tada vrijedi:

$$\vec{a} \circ \vec{b} = a_x \cdot b_x + a_y \cdot b_y \quad , \quad \left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2} \quad , \quad \left|\vec{b}\right| = \sqrt{b_x^2 + b_y^2} \quad , \quad \cos\varphi = \frac{a_x \cdot b_x + a_y \cdot b_y}{\sqrt{a_x^2 + a_y^2} \cdot \sqrt{b_x^2 + b_y^2}},$$

gdje je φ kut koji međusobno zatvaraju vektori $\vec{a}$ i $\vec{b}$.

Budući da je duljina vektora $\vec{AB}$ jednaka duljini dužine $\overline{AB}$, vrijedi:

$$\left|\vec{AB}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2},$$

gdje su $A(x_1, y_1)$, $B(x_2, y_2)$.

a) Prikazi vektora $\vec{OA}$ i $\vec{AB}$ kao linearne kombinacije jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$ glase:

- $$\left.\begin{array}{l} O(x_1, y_1) = O(0, 0) \\ A(x_2, y_2) = A(5, 1) \\ \vec{OA} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{OA} = (5-0)\cdot \vec{i} + (1-0)\cdot \vec{j} \Rightarrow \vec{OA} = 5\cdot \vec{i} + \vec{j}.$$

- $$\left.\begin{array}{l} A(x_1, y_1) = A(5, 1) \\ B(x_2, y_2) = B(6, 3) \\ \vec{AB} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{AB} = (6-5)\cdot \vec{i} + (3-1)\cdot \vec{j} \Rightarrow \vec{AB} = \vec{i} + 2\cdot \vec{j}.$$

Računamo duljine vektora $\vec{OA}$ i $\vec{AB}$.

- $\vec{OA} = 5\cdot \vec{i} + \vec{j} \Rightarrow \left|\vec{OA}\right| = \sqrt{5^2 + 1^2} \Rightarrow \left|\vec{OA}\right| = \sqrt{25+1} \Rightarrow \left|\vec{OA}\right| = \sqrt{26}.$
- $\vec{AB} = \vec{i} + 2\cdot \vec{j} \Rightarrow \left|\vec{AB}\right| = \sqrt{1^2 + 2^2} \Rightarrow \left|\vec{AB}\right| = \sqrt{1+4} \Rightarrow \left|\vec{AB}\right| = \sqrt{5}.$

b) Računamo koordinate točke C(x, y) tako da vrijedi:

$$\vec{OC} = \vec{AB}.$$

Prikazi vektora $\vec{OC}$ i $\vec{AB}$ kao linearne kombinacije jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$ glase:

- $$\left.\begin{array}{l} O(x_1, y_1) = O(0, 0) \\ C(x_2, y_2) = C(x, y) \\ \vec{OC} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{OC} = (x-0)\cdot \vec{i} + (y-0)\cdot \vec{j} \Rightarrow \vec{OC} = x\cdot \vec{i} + y\cdot \vec{j}.$$

- $$\left.\begin{array}{l} A(x_1, y_1) = A(5, 1) \\ B(x_2, y_2) = B(6, 3) \\ \vec{AB} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{AB} = (6-5)\cdot \vec{i} + (3-1)\cdot \vec{j} \Rightarrow \vec{AB} = \vec{i} + 2\cdot \vec{j}.$$

Budući da mora biti

$$\vec{OC} = \vec{AB},$$

slijedi:

$$\left.\begin{array}{l} \vec{OC} = x\cdot \vec{i} + y\cdot \vec{j} \\ \vec{AB} = \vec{i} + 2\cdot \vec{j} \\ \vec{OC} = \vec{AB} \end{array}\right\} \Rightarrow x\cdot \vec{i} + y\cdot \vec{j} = \vec{i} + 2\cdot \vec{j} \Rightarrow \left.\begin{array}{l} x = 1 \\ y = 2 \end{array}\right\} \Rightarrow C(x, y) = C(1, 2).$$

c) Određujemo mjeru kuta $\angle AOC$. Uočimo vektore $\vec{OA}$ i $\vec{OC}$.

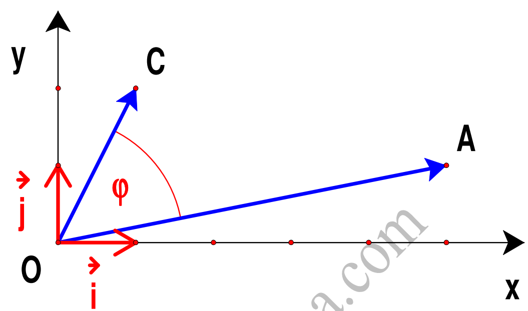
$$\left.\begin{aligned}\vec{OA} &= 5\cdot\vec{i}+\vec{j}\\ \vec{OC} &= \vec{i}+2\cdot\vec{j}\end{aligned}\right\}.$$

Budući da je φ kut između tih vektora, pomoću formule za skalarni produkt dobije se:

$$\cos\varphi=\frac{\vec{OA}\circ\vec{OC}}{\left|\vec{OA}\right|\cdot\left|\vec{OC}\right|}\Rightarrow\cos\varphi=\frac{\left(5\cdot\vec{i}+\vec{j}\right)\circ\left(\vec{i}+2\cdot\vec{j}\right)}{\left|5\cdot\vec{i}+\vec{j}\right|\cdot\left|\vec{i}+2\cdot\vec{j}\right|}\Rightarrow\cos\varphi=\frac{5\cdot1+1\cdot2}{\sqrt{5^2+1^2}\cdot\sqrt{1^2+2^2}}\Rightarrow$$

$$\Rightarrow\cos\varphi=\frac{5+2}{\sqrt{25+1}\cdot\sqrt{1+4}}\Rightarrow\cos\varphi=\frac{7}{\sqrt{26}\cdot\sqrt{5}}\Rightarrow\cos\varphi=\frac{7}{\sqrt{26\cdot5}}\Rightarrow\cos\varphi=\frac{7}{\sqrt{130}}\Rightarrow$$

$$\Rightarrow\varphi=\cos^{-1}\left(\frac{7}{\sqrt{130}}\right)\Rightarrow\varphi=52^0.$$



Vježba 049

U koordinatnom sustavu zadane su točke A(6, 2) i B(7, 4).

Prikaži vektor $\vec{AB}$ kao linearnu kombinaciju jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$.

Rezultat: $\vec{AB}=\vec{i}+2\cdot\vec{j}$.

Zadatak 050 (Sanja, gimnazija)

Odredi parametar λ tako da vektori $\vec{a}=(\lambda-1)\cdot\vec{p}+(2\cdot\lambda+1)\cdot\vec{q}$ i $\vec{b}=-3\cdot\vec{p}+2\cdot\vec{q}$ budu okomiti, ako je $\left|\vec{p}\right|=1, \left|\vec{q}\right|=2, \angle\left(\vec{p},\vec{q}\right)=120^0$.

Rješenje 050

Ponovimo!

$$\vec{x}\circ\vec{x}=\vec{x}^2\quad,\quad\vec{x}\circ\vec{x}=\left|\vec{x}\right|^2\quad,\quad\vec{x}\circ\vec{y}=\left|\vec{x}\right|\cdot\left|\vec{y}\right|\cdot\cos\varphi\ ,\ \varphi=\angle\left(\vec{x},\vec{y}\right),$$

gdje je φ kut koji međusobno zatvaraju vektori $\vec{x}$ i $\vec{y}$.

$$\vec{x}\circ\vec{y}=\vec{y}\circ\vec{x}.$$

Računamo.

$$\vec{p}^2=\left|\vec{p}\right|^2=1^2=1\quad,\quad\vec{q}^2=\left|\vec{q}\right|^2=2^2=4\quad,\quad\vec{p}\circ\vec{q}=\left|\vec{p}\right|\cdot\left|\vec{q}\right|\cdot\cos\angle\left(\vec{p},\vec{q}\right)=$$

$$=1\cdot2\cdot\cos120^0=1\cdot2\cdot\left(-\frac{1}{2}\right)=-1.$$

Budući da su vektori $\vec{a}$ i $\vec{b}$ međusobno okomiti, njihov je skalarni produkt jednak nuli pa slijedi:

$$\vec{a} \circ \vec{b} = 0 \Rightarrow \left((\lambda-1)\cdot\vec{p}+(2\cdot\lambda+1)\cdot\vec{q}\right)\circ\left(-3\cdot\vec{p}+2\cdot\vec{q}\right)=0 \Rightarrow$$

$$\Rightarrow -3\cdot(\lambda-1)\cdot\vec{p}^{\,2}+2\cdot(\lambda-1)\cdot\vec{p}\circ\vec{q}-3\cdot(2\cdot\lambda+1)\cdot\vec{q}\circ\vec{p}+2\cdot(2\cdot\lambda+1)\cdot\vec{q}^{\,2}=0 \Rightarrow$$

$$\Rightarrow -3\cdot(\lambda-1)\cdot\left|\vec{p}\right|^2+2\cdot(\lambda-1)\cdot\vec{p}\circ\vec{q}-3\cdot(2\cdot\lambda+1)\cdot\vec{p}\circ\vec{q}+2\cdot(2\cdot\lambda+1)\cdot\left|\vec{q}\right|^2=0 \Rightarrow$$

$$\Rightarrow -3\cdot(\lambda-1)\cdot1+2\cdot(\lambda-1)\cdot(-1)-3\cdot(2\cdot\lambda+1)\cdot(-1)+2\cdot(2\cdot\lambda+1)\cdot4=0 \Rightarrow$$

$$\Rightarrow -3\cdot(\lambda-1)-2\cdot(\lambda-1)+3\cdot(2\cdot\lambda+1)+8\cdot(2\cdot\lambda+1)=0 \Rightarrow -5\cdot(\lambda-1)+11\cdot(2\cdot\lambda+1)=0 \Rightarrow$$

$$\Rightarrow -5\cdot\lambda+5+22\cdot\lambda+11=0 \Rightarrow 17\cdot\lambda+16=0 \Rightarrow 17\cdot\lambda=-16 \Rightarrow 17\cdot\lambda=-16\;/:17 \Rightarrow \lambda=-\frac{16}{17}.$$

Vježba 050

Odredi parametar λ tako da vektori $\vec{a}=-(1-\lambda)\cdot\vec{p}+(2\cdot\lambda+1)\cdot\vec{q}$ i $\vec{b}=-3\cdot\vec{p}+2\cdot\vec{q}$ budu okomiti, ako je $\left|\vec{p}\right|=1, \left|\vec{q}\right|=2, \angle\left(\vec{p},\vec{q}\right)=240^0.$

Rezultat: $\lambda=-\frac{16}{17}.$

Zadatak 051 (Sanja, gimnazija)

Izračunati kut $\varphi=\angle\left(\vec{a},\vec{b}\right)$, ako je $\left|\vec{a}\right|=2, \left|\vec{b}\right|=3$ i $\left|\vec{a}+2\cdot\vec{b}\right|=\left|2\cdot\vec{a}-\vec{b}\right|.$

Rješenje 051

Ponovimo!

$$\left(\sqrt{x}\right)^2=x \quad , \quad \vec{x}\circ\vec{x}=\vec{x}^{\,2} \quad , \quad \vec{x}\circ\vec{x}=\left|\vec{x}\right|^2 \quad , \quad \vec{x}\circ\vec{y}=\left|\vec{x}\right|\cdot\left|\vec{y}\right|\cdot\cos\varphi \;,\; \varphi=\angle\left(\vec{x},\vec{y}\right),$$

gdje je φ kut koji međusobno zatvaraju vektori $\vec{x}$ i $\vec{y}$.

$$\left|\vec{x}\right|=\sqrt{\vec{x}\circ\vec{x}} \quad , \quad \cos\varphi=\frac{\vec{x}\circ\vec{y}}{\left|\vec{x}\right|\cdot\left|\vec{y}\right|} \quad , \quad (x+y)^2=x^2+2\cdot x\cdot y+y^2 \quad , \quad (x-y)^2=x^2-2\cdot x\cdot y+y^2.$$

Skalarni produkt vektora $\vec{a}\circ\vec{b}$ nalazimo iz uvjeta zadatka:

$$\left|\vec{a}+2\cdot\vec{b}\right|=\left|2\cdot\vec{a}-\vec{b}\right| \Rightarrow \sqrt{\left(\vec{a}+2\cdot\vec{b}\right)^2}=\sqrt{\left(2\cdot\vec{a}-\vec{b}\right)^2} \Rightarrow \sqrt{\left(\vec{a}+2\cdot\vec{b}\right)^2}=\sqrt{\left(2\cdot\vec{a}-\vec{b}\right)^2}\;/^2 \Rightarrow$$

$$\Rightarrow \left(\sqrt{\left(\vec{a}+2\cdot\vec{b}\right)^2}\right)^2=\left(\sqrt{\left(2\cdot\vec{a}-\vec{b}\right)^2}\right)^2 \Rightarrow \left(\vec{a}+2\cdot\vec{b}\right)^2=\left(2\cdot\vec{a}-\vec{b}\right)^2 \Rightarrow$$

$$\Rightarrow \vec{a}^{\,2}+4\cdot\vec{a}\circ\vec{b}+4\cdot\vec{b}^{\,2}=4\cdot\vec{a}^{\,2}-4\cdot\vec{a}\circ\vec{b}+\vec{b}^{\,2} \Rightarrow$$

$$\Rightarrow \left|\vec{a}\right|^2+4\cdot\vec{a}\circ\vec{b}+4\cdot\left|\vec{b}\right|^2=4\cdot\left|\vec{a}\right|^2-4\cdot\vec{a}\circ\vec{b}+\left|\vec{b}\right|^2 \Rightarrow$$

$$\Rightarrow 2^2+4\cdot \vec{a}\circ\vec{b}+4\cdot 3^2=4\cdot 2^2-4\cdot \vec{a}\circ\vec{b}+3^2 \Rightarrow 4+4\cdot \vec{a}\circ\vec{b}+4\cdot 9=4\cdot 4-4\cdot \vec{a}\circ\vec{b}+9 \Rightarrow$$

$$\Rightarrow 4+4\cdot \vec{a}\circ\vec{b}+36=16-4\cdot \vec{a}\circ\vec{b}+9 \Rightarrow 4\cdot \vec{a}\circ\vec{b}+4\cdot \vec{a}\circ\vec{b}=16+9-4-36 \Rightarrow$$

$$\Rightarrow 8\cdot \vec{a}\circ\vec{b}=-15 \Rightarrow 8\cdot \vec{a}\circ\vec{b}=-15 \;/:8 \Rightarrow \vec{a}\circ\vec{b}=-\frac{15}{8}.$$

Sada računamo kut φ.

$$\cos\varphi=\frac{\vec{a}\circ\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|} \Rightarrow \cos\varphi=\frac{-\frac{15}{8}}{2\cdot 3} \Rightarrow \cos\varphi=\frac{-\frac{15}{8}}{6} \Rightarrow \cos\varphi=\frac{-\frac{15}{8}}{\frac{6}{1}} \Rightarrow \cos\varphi=\frac{-\frac{5}{8}}{\frac{2}{1}} \Rightarrow$$

$$\Rightarrow \cos\varphi=-\frac{5}{16} \Rightarrow \varphi=\cos^{-1}\left(-\frac{5}{16}\right) \Rightarrow \varphi=108^0 12'36''.$$

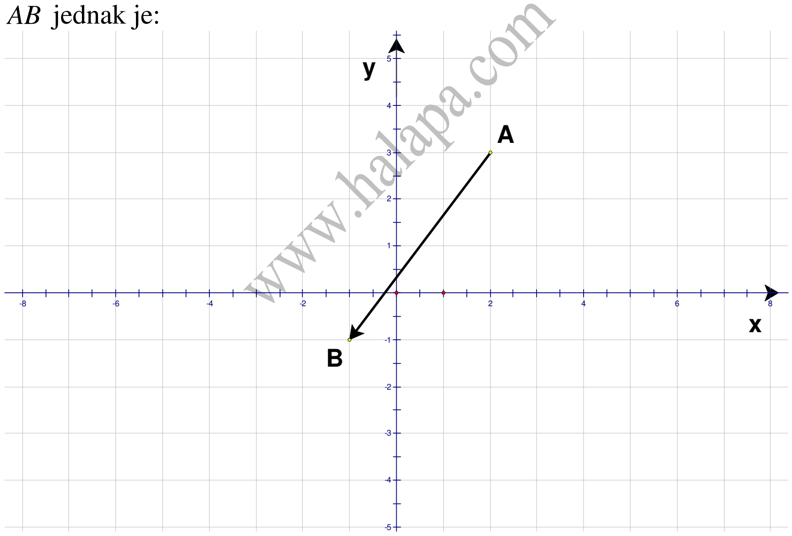
Vježba 051

Izračunati kut $\varphi=\angle\left(\vec{a},\vec{b}\right)$, ako je $\left|\vec{a}\right|=2$, $\left|\vec{b}\right|=3$ i $\left|2\cdot\vec{b}+\vec{a}\right|=\left|\vec{b}-2\cdot\vec{a}\right|$.

Rezultat: $\varphi=108^0 12'36''.$

Zadatak 052 (Matija, Josipa, TUPŠ)

Vektor $\overrightarrow{AB}$ jednak je:



$A.\ \overrightarrow{AB}=-3\cdot\vec{i}-4\cdot\vec{j}$ $\quad$ $B.\ \overrightarrow{AB}=-4\cdot\vec{i}-3\cdot\vec{j}$

$C.\ \overrightarrow{AB}=3\cdot\vec{i}-4\cdot\vec{j}$ $\quad$ $D.\ \overrightarrow{AB}=-4\cdot\vec{i}+3\cdot\vec{j}$

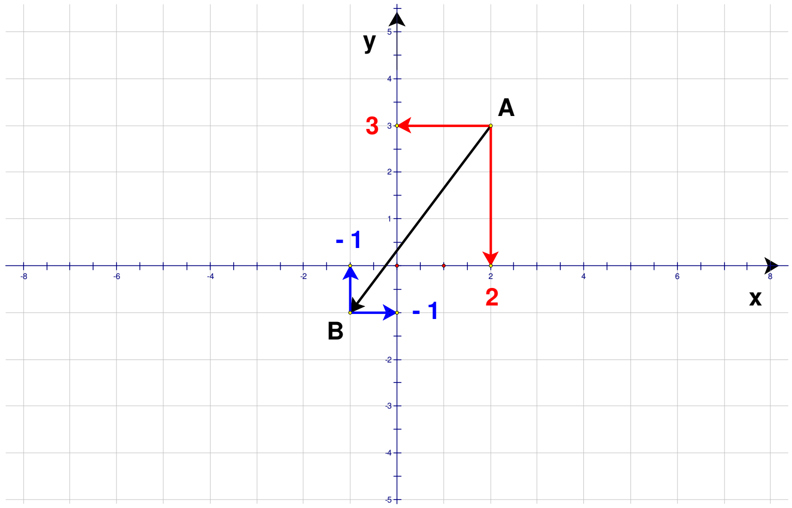
Rješenje 052

Ponovimo!

Ako su dane točke $A(x_1, y_1)$ i $B(x_2, y_2)$, onda su koordinate vektora koji ih spaja:

$$\overrightarrow{AB}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j}.$$

Sa slike očitaju se koordinate točke A (koja je početna točka vektora) i točke B (koja je završna točka vektora).



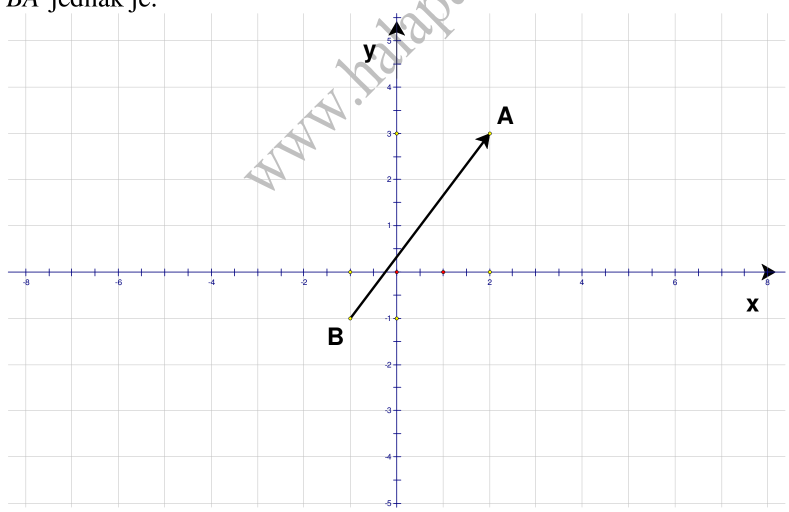
$$\left.\begin{array}{l} A\left(x_1,\ y_1\right)=A(2,\ 3)\ ,\ B\left(x_2,\ y_2\right)=B(-1,\ -1) \\ \overrightarrow{AB}=\left(x_2-x_1\right)\cdot\ \overrightarrow{i}+\left(y_2-y_1\right)\cdot\ \overrightarrow{j} \end{array}\right\}\Rightarrow\ \overrightarrow{AB}=(-1-2)\cdot\ \overrightarrow{i}+(-1-3)\cdot\ \overrightarrow{j}\ \Rightarrow$$

$$\Rightarrow\ \overrightarrow{AB}=-3\cdot\ \overrightarrow{i}-4\cdot\ \overrightarrow{j}\ .$$

Odgovor je pod A.

Vježba 052

Vektor $\overrightarrow{BA}$ jednak je:



$A.\ \overrightarrow{BA}=3\cdot\ \overrightarrow{i}+4\cdot\ \overrightarrow{j}$ $\qquad B.\ \overrightarrow{BA}=4\cdot\ \overrightarrow{i}+3\cdot\ \overrightarrow{j}$

$C.\ \overrightarrow{BA}=-3\cdot\ \overrightarrow{i}+4\cdot\ \overrightarrow{j}$ $\qquad D.\ \overrightarrow{BA}=4\cdot\ \overrightarrow{i}-3\cdot\ \overrightarrow{j}$

Rezultat: A.

Zadatak 053 (Pavle, gimnazija)

Odredite realan broj k tako da vektori $\overrightarrow{a}=6\cdot\ \overrightarrow{i}-4\cdot\ \overrightarrow{j}$ i $\overrightarrow{b}=2\cdot\ \overrightarrow{i}+(2\cdot k+5)\cdot\ \overrightarrow{j}$ budu okomiti.

Rješenje 053

Ponovimo!

Ako su vektori zadani u koordinatnom sustavu, tada se skalarni produkt definira na ovaj način:

$$\left.\begin{array}{l}\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}\\ \vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}\end{array}\right\}\Rightarrow\ \vec{a}\circ\vec{b}=a_x\cdot b_x+a_y\cdot b_y.$$

Dva su vektora $\vec{a}$ i $\vec{b}$ okomita ako im je skalarni produkt jednak nuli:

$$\vec{a}\perp\vec{b}\ \Rightarrow\ a_x\cdot b_x+a_y\cdot b_y=0.$$

Računamo parametar k.

$$\left.\begin{array}{l}\vec{a}=6\cdot\vec{i}-4\cdot\vec{j}\\ \vec{b}=2\cdot\vec{i}+(2\cdot k+5)\cdot\vec{j}\\ \vec{a}\perp\vec{b}\end{array}\right\}\Rightarrow\ \vec{a}\circ\vec{b}=0\ \Rightarrow\ \left(6\cdot\vec{i}-4\cdot\vec{j}\right)\circ\left(2\cdot\vec{i}+(2\cdot k+5)\cdot\vec{j}\right)=0\ \Rightarrow$$

$$\Rightarrow\ 6\cdot2+(-4)\cdot(2\cdot k+5)=0\ \Rightarrow\ 12-8\cdot k-20=0\ \Rightarrow\ -8\cdot k=-12+20\ \Rightarrow\ -8\cdot k=8\ \Rightarrow$$

$$\Rightarrow\ -8\cdot k=8\ /:(-8)\ \Rightarrow\ k=-1.$$

Vježba 053

Odredite realan broj k tako da vektori $\vec{a}=3\cdot\vec{i}-2\cdot\vec{j}$ i $\vec{b}=2\cdot\vec{i}+(2\cdot k+5)\cdot\vec{j}$ budu okomiti.

Rezultat: k = – 1.

Zadatak 054 (Vedra Tea ☺, gimnazija)

Zadani su vektori $\vec{a}=(1,\,-2,\,1)$, $\vec{b}=(-1,\,1,\,2)$ i $\vec{c}=(1,\,y,\,z)$. Ako je $\vec{c}\perp\vec{a}$ i $\vec{c}\perp\vec{b}$, izračunajte y i z.

Rješenje 054

Ponovimo!

Ako su vektori zadani u koordinatnom sustavu, tada se skalarni produkt definira na ovaj način:

$$\left.\begin{array}{l}\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}\\ \vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}\end{array}\right\}\Rightarrow\ \vec{a}\circ\vec{b}=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z.$$

Dva su vektora $\vec{a}$ i $\vec{b}$ okomita ako im je skalarni produkt jednak nuli:

$$\vec{a}\perp\vec{b}\ \Rightarrow\ a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z=0.$$

Računamo y i z.

$$\left.\begin{array}{l}\vec{c}\perp\vec{a}\\ \vec{c}\perp\vec{b}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\vec{c}\circ\vec{a}=0\\ \vec{c}\circ\vec{b}=0\end{array}\right\}\Rightarrow\left[\begin{array}{l}\vec{c}=(1,\,y,\,z)\\ \vec{a}=(1,\,-2,\,1)\\ \vec{b}=(-1,\,1,\,2)\end{array}\right]\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}1\cdot1+y\cdot(-2)+z\cdot1=0\\ 1\cdot(-1)+y\cdot1+z\cdot2=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}1-2\cdot y+z=0\\ -1+y+2\cdot z=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}-2\cdot y+z=-1\\ y+2\cdot z=1\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{metoda suprotnih}\\ \text{koeficijenata}\end{array}\right]\Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} -2\cdot y+z=-1 \\ y+2\cdot z=1 \;/\cdot 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -2\cdot y+z=-1 \\ 2\cdot y+4\cdot z=2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -2\cdot y+z=-1 \\ 2\cdot y+4\cdot z=2 \end{array}\right\} \Rightarrow 5\cdot z=1 \Rightarrow 5\cdot z=1 \;/:5 \Rightarrow z=\frac{1}{5}.$$

Računamo y.

$$\left.\begin{array}{l} y+2\cdot z=1 \\ z=\frac{1}{5} \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{supstitucije} \end{array}\right] \Rightarrow y+2\cdot\frac{1}{5}=1 \Rightarrow y+\frac{2}{5}=1 \Rightarrow y=1-\frac{2}{5} \Rightarrow$$

$$\Rightarrow y=\frac{1}{1}-\frac{2}{5} \Rightarrow y=\frac{5-2}{5} \Rightarrow y=\frac{3}{5}.$$

Vježba 054

Zadani su vektori $\vec{a}=(1,\ y,\ z)$, $\vec{b}=(-1,\ 1,\ 2)$ i $\vec{c}=(1,\ -2,\ 1)$. Ako je $\vec{a}\perp\vec{b}$ i $\vec{a}\perp\vec{c}$, izračunajte y i z.

Rezultat: $y=\frac{3}{5},\ z=\frac{1}{5}.$

Zadatak 055 (Mario, gimnazija)

Zadane su točke M(2, 3), N(– 1, 4) i P(7, – 3). Vektor $\overrightarrow{MN}+\overrightarrow{MP}$ prikaži kao linearnu kombinaciju jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$.

Rješenje 055

Ponovimo!

Ako su dane točke A(x_1, y_1) i B(x_2, y_2), onda su koordinate vektora koji ih spaja:

$$\overrightarrow{AB}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j}.$$

Najprije odredimo vektore.

- Vektor $\overrightarrow{MN}$

$$\left.\begin{array}{l} M\left(x_1,\ y_1\right)=M(2,\ 3) \\ N\left(x_2,\ y_2\right)=N(-1,\ 4) \\ \overrightarrow{MN}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j} \end{array}\right\} \Rightarrow \overrightarrow{MN}=(-1-2)\cdot\vec{i}+(4-3)\cdot\vec{j} \Rightarrow \overrightarrow{MN}=-3\cdot\vec{i}+\vec{j}.$$

- Vektor $\overrightarrow{MP}$

$$\left.\begin{array}{l} M\left(x_1,\ y_1\right)=M(2,\ 3) \\ P\left(x_2,\ y_2\right)=P(7,\ -3) \\ \overrightarrow{MP}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j} \end{array}\right\} \Rightarrow \overrightarrow{MP}=(7-2)\cdot\vec{i}+(-3-3)\cdot\vec{j} \Rightarrow \overrightarrow{MP}=5\cdot\vec{i}-6\cdot\vec{j}.$$

Sada je:

$$\overrightarrow{MN}+\overrightarrow{MP}=-3\cdot\vec{i}+\vec{j}+5\cdot\vec{i}-6\cdot\vec{j}=2\cdot\vec{i}-5\cdot\vec{j}.$$

Vježba 055

Zadane su točke M(2, 3), N(– 1, 4) i P(7, – 3). Vektor $\overrightarrow{MN}-\overrightarrow{MP}$ prikaži kao linearnu kombinaciju jediničnih okomitih vektora $\vec{i}$ i $\vec{j}$.

Rezultat: $\overrightarrow{MN}-\overrightarrow{MP}=-8\cdot\vec{i}+7\cdot\vec{j}$.

Zadatak 056 (Josip, gimnazija)

Dane su točke A(– 1, – 3), B(2, – 1), C(0, 4), D(1, y). Vektori $\overrightarrow{AB}$ i $\overrightarrow{CD}$ su okomiti ako je

$$A.\ y=\frac{1}{2}\qquad B.\ y=\frac{3}{2}\qquad C.\ y=\frac{5}{2}\qquad D.\ y=\frac{7}{2}$$

Rješenje 056

Ponovimo!

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c\quad,\quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Neka su $A(x_1, y_1)$ i $B(x_2, y_2)$ dvije točke ravnine. Tada vrijedi: $\overrightarrow{AB}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}$.

Ako su vektori zadani u koordinatnom sustavu, tada se skalarni produkt definira na ovaj način:

$$\left.\begin{array}{l}\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}\\ \vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}\end{array}\right\}\Rightarrow\ \vec{a}\circ\vec{b}=a_x\cdot b_x+a_y\cdot b_y.$$

Dva su vektora $\vec{a}$ i $\vec{b}$ okomita ako im je skalarni produkt jednak nuli:

$$\vec{a}\perp\vec{b}\ \Rightarrow\ a_x\cdot b_x+a_y\cdot b_y=0.$$

Budući da su zadane točke A(– 1, – 3), B(2, – 1), C(0, 4), D(1, y) vektori $\overrightarrow{AB}$ i $\overrightarrow{CD}$ glase:

- $$\left.\begin{array}{l}A(x_1, y_1)=A(-1,-3)\\ B(x_2, y_2)=B(2,-1)\\ \overrightarrow{AB}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\end{array}\right\}\Rightarrow\ \overrightarrow{AB}=(2-(-1))\cdot\vec{i}+(-1-(-3))\cdot\vec{j}\ \Rightarrow$$

$$\Rightarrow\ \overrightarrow{AB}=(2+1)\cdot\vec{i}+(-1+3)\cdot\vec{j}\ \Rightarrow\ \overrightarrow{AB}=3\cdot\vec{i}+2\cdot\vec{j}.$$

- $$\left.\begin{array}{l}C(x_1, y_1)=C(0,4)\\ D(x_2, y_2)=D(1,y)\\ \overrightarrow{CD}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\end{array}\right\}\Rightarrow\ \overrightarrow{CD}=(1-0)\cdot\vec{i}+(y-4)\cdot\vec{j}\ \Rightarrow\ \overrightarrow{CD}=\vec{i}+(y-4)\cdot\vec{j}.$$

Vektori $\overrightarrow{AB}$ i $\overrightarrow{CD}$ su okomiti pa je njihov skalarni produkt jednak nuli.

$$\left.\begin{array}{l}\overrightarrow{AB}=a_x\cdot\vec{i}+a_y\cdot\vec{j}\ ,\ \overrightarrow{AB}=3\cdot\vec{i}+2\cdot\vec{j}\\ \overrightarrow{CD}=b_x\cdot\vec{i}+b_y\cdot\vec{j}\ ,\ \overrightarrow{CD}=\vec{i}+(y-4)\cdot\vec{j}\end{array}\right\}\Rightarrow\left[\begin{array}{c}\text{uvjet okomitosti}\\ a_x\cdot b_x+a_y\cdot b_y=0\end{array}\right]\Rightarrow$$

$$\Rightarrow\ 3\cdot 1+2\cdot(y-4)=0\ \Rightarrow\ 3+2\cdot y-8=0\ \Rightarrow\ 2\cdot y=-3+8\ \Rightarrow\ 2\cdot y=5\ \Rightarrow\ 2\cdot y=5\ /:2\ \Rightarrow\ y=\frac{5}{2}.$$

Odgovor je pod C.

Vježba 056

Dane su točke A(– 1, – 3), B(2, – 1), C(0, 4), D(2, y). Vektori $\overrightarrow{AB}$ i $\overrightarrow{CD}$ su okomiti ako je

$$A.\ y=1 \qquad B.\ y=2 \qquad C.\ y=3 \qquad D.\ y=4$$

Rezultat: A.

Zadatak 057 (Josip, gimnazija)

Kut između vektora $\vec{a}$ i $\vec{b}$ iznosi 120º. Ako je $\left|\vec{a}\right|=4$ i $\left|\vec{b}\right|=5,$ onda je

$$A.\ \left|\vec{a}+\vec{b}\right|=12 \qquad B.\ \left|\vec{a}+\vec{b}\right|=4\cdot\sqrt{7} \qquad C.\ \left|\vec{a}+\vec{b}\right|=10 \qquad D.\ \left|\vec{a}+\vec{b}\right|=\sqrt{21}$$

Rješenje 057

Ponovimo!

$$\cos 120^0=-\frac{1}{2}.$$

Skalarni produkt vektora $\vec{a}$ i $\vec{b}$: $\vec{a}\circ\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha$, gdje je α kut između vektora $\vec{a}$ i $\vec{b}$.

$$\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos 0=\left|\vec{a}\right|^2\cdot 1=\left|\vec{a}\right|^2 \Rightarrow \left|\vec{a}\right|=\sqrt{\vec{a}\circ\vec{a}}.$$

$$\left(\vec{a}+\vec{b}\right)\circ\left(\vec{a}+\vec{b}\right)=\vec{a}\circ\vec{a}+\vec{a}\circ\vec{b}+\vec{b}\circ\vec{a}+\vec{b}\circ\vec{b}=\left|\vec{a}\right|^2+2\cdot\vec{a}\circ\vec{b}+\left|\vec{b}\right|^2.$$

$$\left|\vec{a}+\vec{b}\right|=\sqrt{\left(\vec{a}+\vec{b}\right)\circ\left(\vec{a}+\vec{b}\right)} \Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{\vec{a}\circ\vec{a}+\vec{a}\circ\vec{b}+\vec{b}\circ\vec{a}+\vec{b}\circ\vec{b}} \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{\left|\vec{a}\right|^2+2\cdot\vec{a}\circ\vec{b}+\left|\vec{b}\right|^2} \Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{\left|\vec{a}\right|^2+2\cdot\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha+\left|\vec{b}\right|^2} \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{4^2+2\cdot 4\cdot 5\cdot\cos 120^0+5^2} \Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{16+2\cdot 4\cdot 5\cdot\left(-\frac{1}{2}\right)+25} \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{16+2\cdot 4\cdot 5\cdot\left(-\frac{1}{2}\right)+25} \Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{16-20+25} \Rightarrow \left|\vec{a}+\vec{b}\right|=\sqrt{21}.$$

Odgovor je pod D.

Vježba 057

Kut između vektora $\vec{a}$ i $\vec{b}$ iznosi 60º. Ako je $\left|\vec{a}\right|=4$ i $\left|\vec{b}\right|=5,$ onda je

$$A.\ \left|\vec{a}+\vec{b}\right|=12 \qquad B.\ \left|\vec{a}+\vec{b}\right|=3\cdot\sqrt{7} \qquad C.\ \left|\vec{a}+\vec{b}\right|=25 \qquad D.\ \left|\vec{a}+\vec{b}\right|=\sqrt{61}$$

Rezultat: D.

Zadatak 058 (Gabi, ekonomska škola)

Zadani su vektori $\vec{a}=2\cdot\vec{i}-3\cdot\vec{j},\ \vec{b}=-\vec{i}+2\cdot\vec{j}$. Odredi $\left|\vec{a}+2\cdot\vec{b}\right|.$

Rješenje 058

Ponovimo!

Zakon distribucije množenja prema zbrajanju.

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b+c).$$

Duljina vektora $\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j}$ definira se $\left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2}$.

$$\left|\vec{a} + 2 \cdot \vec{b}\right| = \left|2 \cdot \vec{i} - 3 \cdot \vec{j} + 2 \cdot \left(-\vec{i} + 2 \cdot \vec{j}\right)\right| = \left|2 \cdot \vec{i} - 3 \cdot \vec{j} - 2 \cdot \vec{i} + 4 \cdot \vec{j}\right| = \left|2 \cdot \vec{i} - 3 \cdot \vec{j} - 2 \cdot \vec{i} + 4 \cdot \vec{j}\right| =$$

$$= \left|-3 \cdot \vec{j} + 4 \cdot \vec{j}\right| = \left|\vec{j}\right| = \left|0 \cdot \vec{i} + 1 \cdot \vec{j}\right| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1.$$

Vježba 058

Zadani su vektori $\vec{a} = 4 \cdot \vec{i} - 6 \cdot \vec{j}$, $\vec{b} = -2 \cdot \vec{i} + 4 \cdot \vec{j}$. Odredi $\left|\vec{a} + 2 \cdot \vec{b}\right|$.

Rezultat: 2.

Zadatak 059 (Darko, gimnazija)

Odredite površinu trokuta ABC ako je točka O ishodište koordinatnoga sustava, vektor $\overrightarrow{OA} = -2 \cdot \vec{i} + \vec{j}$, vektor $\overrightarrow{AB} = 5 \cdot \vec{i} - 3 \cdot \vec{j}$, vektor $\overrightarrow{AC}$ je usporedan s vektorom $\vec{i}$, a skalarni umnožak $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$. Napomena: Po potrebi skicu nacrtajte u koordinatnom sustavu.

Rješenje 059

Ponovimo!

Ako je zadana točka A(x, y) pripadni radijus vektor glasi: $\vec{r} = x \cdot \vec{i} + y \cdot \vec{j}$.
Ako su dane točke A(x_1, y_1) i B(x_2, y_2), onda su koordinate vektora koji ih spaja:

$$\overrightarrow{AB} = \left(x_2 - x_1\right) \cdot \vec{i} + \left(y_2 - y_1\right) \cdot \vec{j}.$$

Ako je $a \cdot \vec{i} + b \cdot \vec{j} = c \cdot \vec{i} + d \cdot \vec{j}$, onda je a = c i b = d.
Dva su vektora jednaka ako i samo ako su im odgovarajuće koordinate jednake.
Dva su vektora međusobno okomita ako je njihov skalarni produkt jednak nuli, tj. ako vrijedi

$$\left.\begin{array}{l} \vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} \\ \vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j} \\ \vec{a} \perp \vec{b} \end{array}\right\} \Rightarrow a_x \cdot b_x + a_y \cdot b_y = 0.$$

Zakon distribucije množenja prema zbrajanju.

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b+c).$$

Trokut je dio ravnine omeđen s tri dužine. Te dužine zovemo stranice trokuta.
Pravokutni trokuti imaju jedan pravi kut (kut od 90º). Stranice koje zatvaraju pravi kut zovu se katete, a najdulja stranica je hipotenuza pravokutnog trokuta.
Ploština pravokutnog trokuta čije su katete a i b dana je formulom

$$P = \frac{a \cdot b}{2}.$$

Budući da je duljina vektora $\overrightarrow{AB}$ jednaka duljini dužine $\overline{AB}$, vrijedi:

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2},$$

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}.$$

Površina trokuta ABC zadanog vrhovima $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ računa se po formuli:

$$P_{ABC}=\frac{1}{2}\cdot\left|x_1\cdot(y_2-y_3)+x_2\cdot(y_3-y_1)+x_3\cdot(y_1-y_2)\right|.$$

Za realni broj x njegova je apsolutna vrijednost (modul) broj $|x|$ koji određujemo na ovaj način:

$$|x|=\begin{cases} x, & x\ge 0 \\ -x, & x<0. \end{cases}$$

Ako je broj x pozitivan ili nula, tada je on jednak svojoj apsolutnoj vrijednosti. Za svaki x, $x \ge 0$, vrijedi $|x| = x$.
Ako je x negativan broj, njegova apsolutna vrijednost je suprotan broj – x koji je pozitivan. Za svaki x, $x < 0$, je $|x| = -x$.
Skratiti razlomak znači brojnik i nazivnik tog razlomka podijeliti istim brojem različitim od nule i jedinice

$$\frac{a\cdot n}{b\cdot n}=\frac{a}{b}\ ,\ n\ne 0\ ,\ n\ne 1.$$

Koordinate točke A odredimo iz radijusa vektora $\overrightarrow{OA}$.

$$\left.\begin{matrix}\overrightarrow{OA}=-2\cdot\vec{i}+\vec{j}\\ \overrightarrow{OA}=x\cdot\vec{i}+y\cdot\vec{j}\end{matrix}\right\}\Rightarrow\left.\begin{matrix}\overrightarrow{OA}=-2\cdot\vec{i}+1\cdot\vec{j}\\ \overrightarrow{OA}=x\cdot\vec{i}+y\cdot\vec{j}\end{matrix}\right\}\Rightarrow A(x, y)=A(-2, 1).$$

Koordinate točke B nađemo pomoću vektora $\overrightarrow{AB}$.

$$\left.\begin{matrix}A(x_1, y_1)=A(-2, 1)\\ B(x_2, y_2)=B(x, y)\end{matrix}\right\}\Rightarrow\left[\overrightarrow{AB}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\right]\Rightarrow\overrightarrow{AB}=(x-(-2))\cdot\vec{i}+(y-1)\cdot\vec{j}\Rightarrow$$

$$\Rightarrow\overrightarrow{AB}=(x+2)\cdot\vec{i}+(y-1)\cdot\vec{j}.$$

Dalje slijedi:

$$\left.\begin{matrix}\overrightarrow{AB}=(x+2)\cdot\vec{i}+(y-1)\cdot\vec{j}\\ \overrightarrow{AB}=5\cdot\vec{i}-3\cdot\vec{j}\end{matrix}\right\}\Rightarrow\left.\begin{matrix}x+2=5\\ y-1=-3\end{matrix}\right\}\Rightarrow\left.\begin{matrix}x=5-2\\ y=-3+1\end{matrix}\right\}\Rightarrow\left.\begin{matrix}x=3\\ y=-2\end{matrix}\right\}\Rightarrow B(x, y)=B(3, -2).$$

Budući da je vektor $\overrightarrow{AC}$ usporedan s vektorom $\vec{i}$ (s koordinatnom osi x), koordinate točke C su

$$\left.\begin{matrix}A(x, y)=A(-2, 1)\\ C(x, y)=C(x, 1)\end{matrix}\right\}$$

pa vektor $\overrightarrow{BC}$ glasi:

$$\left.\begin{matrix}B(x_1, y_1)=B(3, -2)\\ C(x_2, y_2)=C(x, 1)\end{matrix}\right\}\Rightarrow\left[\overrightarrow{BC}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\right]\Rightarrow\overrightarrow{BC}=(x-3)\cdot\vec{i}+(1-(-2))\cdot\vec{j}\Rightarrow$$

$$\Rightarrow\overrightarrow{BC}=(x-3)\cdot\vec{i}+(1+2)\cdot\vec{j}\Rightarrow\overrightarrow{BC}=(x-3)\cdot\vec{i}+3\cdot\vec{j}.$$

Koordinate točke C izračunamo iz skalarnog produkta.

$$\left.\begin{matrix}\overrightarrow{AB}=5\cdot\vec{i}-3\cdot\vec{j}\\ \overrightarrow{BC}=(x-3)\cdot\vec{i}+3\cdot\vec{j}\end{matrix}\right\}\Rightarrow\left[\overrightarrow{AB}\circ\overrightarrow{BC}=0\right]\Rightarrow 5\cdot(x-3)-3\cdot 3=0\Rightarrow 5\cdot x-15-9=0\Rightarrow$$

$$\Rightarrow\ 5\cdot x=15+9\ \Rightarrow\ 5\cdot x=24\ \Rightarrow\ 5\cdot x=24\ /:5\ \Rightarrow\ x=4.8.$$

Koordinate točke C su:

$$C(x,\ y)=C(4.8,\ 1).$$

Računamo površinu trokuta ABC.

1.inačica

Zbog skalarnog produkta

$$\overrightarrow{AB}\circ\overrightarrow{BC}=0$$

trokut ABC je pravokutan pa njegova površina glasi:

$$P=\frac{|AB|\cdot|BC|}{2}.$$

Odredimo duljine $|\mathrm{AB}|$ i $|\mathrm{BC}|$.

- $$\left.\begin{array}{l}A(x_1,\ y_1)=A(-2,\ 1)\\ B(x_2,\ y_2)=B(3,\ -2)\end{array}\right\}\Rightarrow\left[\left|\overrightarrow{AB}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right]\Rightarrow$$

$$\Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{(3-(-2))^2+(-2-1)^2}\ \Rightarrow$$

$$\Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{(3+2)^2+(-3)^2}\ \Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{5^2+(-3)^2}\ \Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{25+9}\ \Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{34}$$

- $$\left.\begin{array}{l}B(x_1,\ y_1)=B(3,\ -2)\\ C(x_2,\ y_2)=C(4.8,\ 1)\end{array}\right\}\Rightarrow\left[\left|\overrightarrow{BC}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right]\Rightarrow$$

$$\Rightarrow\left|\overrightarrow{BC}\right|=\sqrt{(4.8-3)^2+(1-(-2))^2}\ \Rightarrow$$

$$\Rightarrow\left|\overrightarrow{BC}\right|=\sqrt{1.8^2+(1+2)^2}\ \Rightarrow\left|\overrightarrow{BC}\right|=\sqrt{1.8^2+3^2}\ \Rightarrow\left|\overrightarrow{BC}\right|=\sqrt{3.24+9}\ \Rightarrow\left|\overrightarrow{BC}\right|=\sqrt{12.24}.$$

Površina trokuta ABC iznosi:

$$P=\frac{|AB|\cdot|BC|}{2}\ \Rightarrow\ P=\frac{\sqrt{34}\cdot\sqrt{12.24}}{2}\ \Rightarrow\ P=\frac{\sqrt{34\cdot12.24}}{2}\ \Rightarrow\ P=\frac{\sqrt{416.16}}{2}\ \Rightarrow\ P=10.2.$$

2.inačica

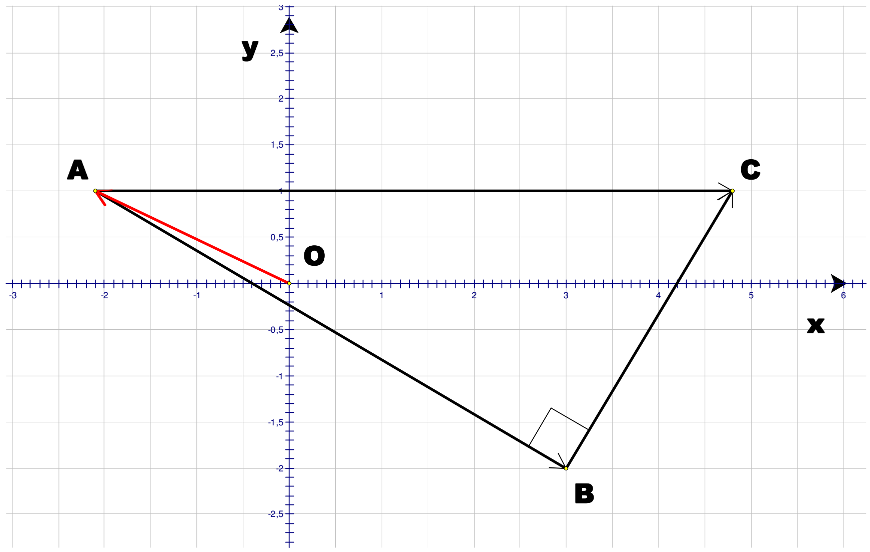
Budući da su zadani vrhovi trokuta ABC njegova površina iznosi:

$$\left.\begin{array}{l}A(x_1,\ y_1)=A(-2,\ 1)\\ B(x_2,\ y_2)=B(3,\ -2)\\ C(x_3,\ y_3)=C(4.8,\ 1)\end{array}\right\}\Rightarrow\left[P_{ABC}=\frac{1}{2}\cdot\left|x_1\cdot(y_2-y_3)+x_2\cdot(y_3-y_1)+x_3\cdot(y_1-y_2)\right|\right]\Rightarrow$$

$$\Rightarrow\ P_{ABC}=\frac{1}{2}\cdot\left|-2\cdot(-2-1)+3\cdot(1-1)+4.8\cdot(1-(-2))\right|\ \Rightarrow\ P_{ABC}=\frac{1}{2}\cdot\left|-2\cdot(-3)+3\cdot0+4.8\cdot(1+2)\right|\ \Rightarrow$$

$$\Rightarrow\ P_{ABC}=\frac{1}{2}\cdot|6+0+4.8\cdot3|\ \Rightarrow\ P_{ABC}=\frac{1}{2}\cdot|6+14.4|\ \Rightarrow\ P_{ABC}=\frac{1}{2}\cdot|20.4|\ \Rightarrow$$

$$\Rightarrow\ P_{ABC}=\frac{1}{2}\cdot20.4\ \Rightarrow\ P_{ABC}=\frac{1}{2}\cdot20.4\ \Rightarrow\ P_{ABC}=10.2.$$



Vježba 059

Odredite opseg trokuta ABC ako je točka O ishodište koordinatnoga sustava, vektor $\overrightarrow{OA} = -2 \cdot \overrightarrow{i} + \overrightarrow{j}$, vektor $\overrightarrow{AB} = 5 \cdot \overrightarrow{i} - 3 \cdot \overrightarrow{j}$, vektor $\overrightarrow{AC}$ je usporedan s vektorom $\overrightarrow{i}$, a skalarni umnožak $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$. Napomena: Po potrebi skicu nacrtajte u koordinatnom sustavu.

Rezultat: 16.13.

Zadatak 060 (Iva, gimnazija)

Zadani su vektori $\overrightarrow{a} = 2 \cdot \overrightarrow{i} + 4 \cdot \overrightarrow{j}$ i $\overrightarrow{b} = 5 \cdot \overrightarrow{i} + k \cdot \overrightarrow{j}$. Odredite sve realne brojeve k za koje je kut između vektora $\overrightarrow{a}$ i $\overrightarrow{b}$ šiljast.

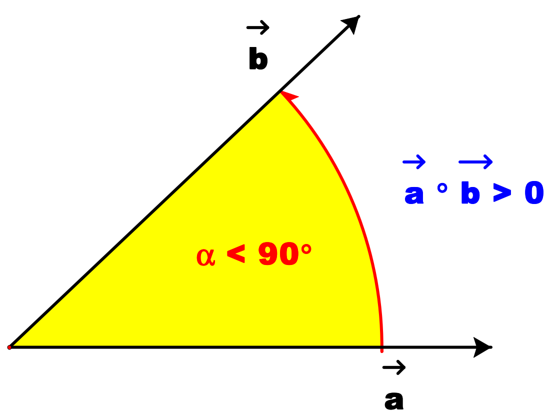
Rješenje 060

Ponovimo!

Ako su $\overrightarrow{a} = a_x \cdot \overrightarrow{i} + a_y \cdot \overrightarrow{j}$ i $\overrightarrow{b} = b_x \cdot \overrightarrow{i} + b_y \cdot \overrightarrow{j}$ dva vektora, tada njihov skalarni produkt glasi:

$$\overrightarrow{a} \circ \overrightarrow{b} = a_x \cdot b_x + a_y \cdot b_y .$$

Kut je skup točaka ravnine određen dvama polupravcima sa zajedničkim početkom. Kutovi koji su manji od pravog kuta zovu se šiljasti ili oštri kutovi.
Kut među vektorima je šiljast ako i samo ako je skalarni produkt veći od nule.



Budući da kut između vektora $\vec{a}$ i $\vec{b}$ mora biti šiljast, skalarni produkt bit će veći od nule pa slijedi:

$$\left.\begin{array}{l} \vec{a} = 2 \cdot \vec{i} + 4 \cdot \vec{j} \\ \vec{b} = 5 \cdot \vec{i} + k \cdot \vec{j} \end{array}\right\} \Rightarrow \left[\begin{array}{l} \vec{a} \circ \vec{b} > 0 \\ a_x \cdot b_x + a_y \cdot b_y > 0 \end{array}\right] \Rightarrow 2 \cdot 5 + 4 \cdot k > 0 \Rightarrow 10 + 4 \cdot k > 0 \Rightarrow$$

$$\Rightarrow 4 \cdot k > -10 \Rightarrow 4 \cdot k > -10 \; /:4 \Rightarrow k > -2.5.$$

Vježba 060

Zadani su vektori $\vec{a} = 5 \cdot \vec{i} + 4 \cdot \vec{j}$ i $\vec{b} = 2 \cdot \vec{i} + k \cdot \vec{j}$. Odredite sve realne brojeve k za koje je kut između vektora $\vec{a}$ i $\vec{b}$ šiljast.

Rezultat: $k > -2.5.$