Zadatak 021 (Hrvoje, šumarska škola)

Zadane su duljine vektore $\vec{a}$ i $\vec{b}$, $\left|\vec{a}\right| = 3$ i $\left|\vec{b}\right| = 4$ i kut među vektorima $\alpha = \frac{2 \cdot \pi}{3}$. Izračunajte $\left(\vec{a} + \vec{b}\right)^2$.

Rješenje 021

$$\left(\vec{a}+\vec{b}\right)^2 = \left(\vec{a}+\vec{b}\right)\circ\left(\vec{a}+\vec{b}\right) = \vec{a}\circ\vec{a}+\vec{a}\circ\vec{b}+\vec{b}\circ\vec{a}+\vec{b}\circ\vec{b} = \vec{a}^{\,2}+2\cdot\vec{a}\circ\vec{b}+\vec{b}^{\,2} =$$

$$= \left|\vec{a}\right|^2+2\cdot\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha+\left|\vec{b}\right|^2 = 3^2+2\cdot3\cdot4\cdot\cos\frac{2\cdot\pi}{3}+4^2 = 9+2\cdot3\cdot4\cdot\left(-\frac{1}{2}\right)+16 = 9-12+16 = 13.$$

Vježba 021

Zadane su duljine vektore $\vec{a}$ i $\vec{b}$, $\left|\vec{a}\right| = 3$ i $\left|\vec{b}\right| = 4$ i kut među vektorima $\alpha = \frac{2 \cdot \pi}{3}$. Izračunajte $\left(\vec{a} - \vec{b}\right)^2$.

Rezultat: 37.

Zadatak 022 (Hrvoje, šumarska škola)

Odredite nepoznate koordinate točaka $A(x, 3)$ i $B(4, y)$ ako je vektor $\overrightarrow{AB} = -3\cdot\vec{i}+2\cdot\vec{j}$.

Rješenje 022

Ponovimo!

Neka su $A(x_1, y_1)$ i $B(x_2, y_2)$ dvije točke ravnine. Tada vrijedi: $\overrightarrow{AB} = (x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}$.

Ako su $\vec{a} = a_x\vec{i} + a_y\vec{j}$, $\vec{b} = b_x\vec{i} + b_y\vec{j}$ dva vektora, oni su jednaki ako i samo ako su im odgovarajuće koordinate jednake, tj. $a_x = b_x$ i $a_y = b_y$.

Računamo nepoznate koordinate x i y:

$$\left.\begin{matrix} A(x,3),\ B(4,y) \\ \overrightarrow{AB} = -3\cdot\vec{i}+2\cdot\vec{j}\end{matrix}\right\} \Rightarrow \left.\begin{matrix} \overrightarrow{AB} = (4-x)\cdot\vec{i}+(y-3)\cdot\vec{j} \\ \overrightarrow{AB} = -3\cdot\vec{i}+2\cdot\vec{j}\end{matrix}\right\} \Rightarrow \left.\begin{matrix} 4-x=-3 \\ y-3=2\end{matrix}\right\} \Rightarrow \left.\begin{matrix} -x=-3-4 \\ y=2+3\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} -x=-7 \ /\cdot(-1) \\ y=5\end{matrix}\right\} \Rightarrow \left.\begin{matrix} x=7 \\ y=5\end{matrix}\right\}.$$

Vježba 022

Odredite nepoznate koordinate točaka $A(x, 3)$ i $B(4, y)$ ako je vektor $\overrightarrow{AB} = 4\cdot\vec{i}-9\cdot\vec{j}$.

Rezultat: x = 0, y = – 6.

Zadatak 023 (Kristina-Kiki, studentica)

Nađite kut α između vektora $\vec{a}$ i $\vec{b}$ ako je $\vec{a}+\vec{b}$ okomito na $7\cdot\vec{a}-5\cdot\vec{b}$ te je $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 2.$

Rješenje 023

Ponovimo!

Skalarni produkt vektora $\vec{a}$ i $\vec{b}$: $\vec{a}\circ\vec{b} = \left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha$,

$\vec{a}\circ\vec{a} = \left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos 0 = \left|\vec{a}\right|^2\cdot 1 = \left|\vec{a}\right|^2$.

Okomitost vektora: $\vec{a}\perp\vec{b} \Rightarrow \vec{a}\circ\vec{b} = 0$.

Budući da je $\vec{a}+\vec{b}$ okomito na $7\cdot\vec{a}-5\cdot\vec{b}$, njihov skalarni produkt jednak je nuli:

$$\left(\vec{a}+\vec{b}\right)\circ\left(7\cdot\vec{a}-5\cdot\vec{b}\right)=0 \Rightarrow 7\cdot\vec{a}\circ\vec{a}-5\cdot\vec{a}\circ\vec{b}+7\cdot\vec{a}\circ\vec{b}-5\cdot\vec{b}\circ\vec{b}=0 \Rightarrow$$

$$\Rightarrow 7\cdot\left|\vec{a}\right|^2+2\cdot\vec{a}\circ\vec{b}-5\cdot\left|\vec{b}\right|^2=0 \Rightarrow 7\cdot 2^2+2\cdot\vec{a}\circ\vec{b}-5\cdot 2^2=0 \Rightarrow 28+2\cdot\vec{a}\circ\vec{b}-20=0 \Rightarrow$$

$$\Rightarrow 8+2\cdot\vec{a}\circ\vec{b}=0 \;/:2 \Rightarrow 4+\vec{a}\circ\vec{b}=0 \Rightarrow \vec{a}\circ\vec{b}=-4 \Rightarrow \left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha=-4 \Rightarrow$$

$$\Rightarrow 2\cdot 2\cdot\cos\alpha=-4 \Rightarrow 4\cdot\cos\alpha=-4 \;/:4 \Rightarrow \cos\alpha=-1 \Rightarrow \alpha=180^0 \text{ ili } \alpha=\pi.$$

Vježba 023

Nađite kut α između vektora $\vec{a}$ i $\vec{b}$ ako je $\vec{a}+\vec{b}$ okomito na $5\cdot\vec{a}-7\cdot\vec{b}$ te je $\left|\vec{a}\right|=2, \left|\vec{b}\right|=2.$

Rezultat: $\alpha=180^0$ ili $\alpha=\pi.$

Zadatak 024 (Hrvoje, šumarska škola)

Za koji je vektor parabola $y=\frac{1}{2}\cdot x^2-12\cdot x+74$ dobivena translacijom iz parabole $y=\frac{1}{2}\cdot x^2-2$?

Rješenje 024

Ponovimo!

Za parabolu $y=a\cdot x^2+b\cdot x+c$ njezino tjeme glasi: $T\left(x_0, y_0\right)=T\left(-\frac{b}{2\cdot a}, \frac{4\cdot a\cdot c-b^2}{4\cdot a}\right).$

Neka su $A\left(x_1, y_1\right)$ i $B\left(x_2, y_2\right)$ dvije točke ravnine. Tada vrijedi: $\overrightarrow{AB}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j}.$

Odredimo tjemena parabola:

$$\left.\begin{array}{l} y=\frac{1}{2}\cdot x^2-2 \\ a=\frac{1}{2}\;,\; b=0\;,\; c=-2 \end{array}\right\} \Rightarrow T_1\left(\frac{0}{2\cdot\frac{1}{2}}, \frac{4\cdot\frac{1}{2}\cdot(-2)-0}{4\cdot\frac{1}{2}}\right) \Rightarrow T_1\left(0, \frac{-4}{2}\right) \Rightarrow T_1(0,-2).$$

$$\left.\begin{array}{l} y=\frac{1}{2}\cdot x^2-12\cdot x+74 \\ a=\frac{1}{2}\;,\; b=-12\;,\; c=74 \end{array}\right\} \Rightarrow T_2\left(\frac{12}{2\cdot\frac{1}{2}}, \frac{4\cdot\frac{1}{2}\cdot 74-144}{4\cdot\frac{1}{2}}\right) \Rightarrow T_2\left(12, \frac{148-144}{2}\right) \Rightarrow T_2(12, 2).$$

Računamo vektore $\overrightarrow{OT_1}$ i $\overrightarrow{OT_2}$:

$$\left.\begin{array}{l} O(0,0) \\ T_1(0,-2) \end{array}\right\} \Rightarrow \overrightarrow{OT_1}=(0-0)\cdot\vec{i}+(-2-0)\cdot\vec{j} \Rightarrow \overrightarrow{OT_1}=-2\cdot\vec{j}.$$

$$\left.\begin{array}{l} O(0,0) \\ T_2(12,2) \end{array}\right\} \Rightarrow \overrightarrow{OT_2}=(12-0)\cdot\vec{i}+(2-0)\cdot\vec{j} \Rightarrow \overrightarrow{OT_2}=12\cdot\vec{i}+2\cdot\vec{j}.$$

Vektor translacije iznosi:

$$\vec{v}=\overrightarrow{OT_2}-\overrightarrow{OT_1}=12\cdot\vec{i}+2\cdot\vec{j}-(-2)\cdot\vec{j}=12\cdot\vec{i}+4\cdot\vec{j}.$$

Vježba 024

Za koji je vektor parabola $y=x^2+4$ dobivena translacijom iz parabole $y=x^2$?

Rezultat: $\vec{v}=4\cdot\vec{j}.$

Zadatak 025 (Ivica, student)

Za vektore $\vec{a}=(2,1,-3)$ i $\vec{b}=(1,1,-2)$ nađite $\left|3\cdot\vec{a}+\vec{b}\right|$.

Rješenje 025

Ponovimo!

Duljina vektora $\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$ definira se $\left|\vec{a}\right|=\sqrt{a_x^2+a_y^2+a_z^2}$.

Računamo:

$$\left.\begin{array}{l}\vec{a}=(2,1,-3)=2\cdot\vec{i}+\vec{j}-3\cdot\vec{k}\\ \vec{b}=(1,1,-2)=\vec{i}+\vec{j}-2\cdot\vec{k}\end{array}\right\}\Rightarrow\left|3\cdot\vec{a}+\vec{b}\right|=\left|3\cdot\left(2\cdot\vec{i}+\vec{j}-3\cdot\vec{k}\right)+\vec{i}+\vec{j}-2\cdot\vec{k}\right|=$$

$$=\left|6\cdot\vec{i}+3\cdot\vec{j}-9\cdot\vec{k}+\vec{i}+\vec{j}-2\cdot\vec{k}\right|=\left|7\cdot\vec{i}+4\cdot\vec{j}-11\cdot\vec{k}\right|=\sqrt{7^2+4^2+(-11)^2}=\sqrt{49+14+121}=\sqrt{186}.$$

Vježba 025

Za vektore $\vec{a}=(1,1,1)$ i $\vec{b}=(2,2,2)$ nađite $\left|\vec{a}+\vec{b}\right|$.

Rezultat: $\sqrt{27}=3\cdot\sqrt{3}.$

Zadatak 026 (Ivica, student)

Za vektore $\vec{b}=(1,1,-2)$ i $\vec{c}=(2,-1,3)$ nađite $\vec{c}\circ\vec{b}$.

Rješenje 026

Ponovimo!

Formula za skalarni produkt vektora $\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$ i $\vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}$ pomoću njihovih komponenata u pravokutnom koordinatnom sustavu (Kartezijevom koordinatnom sustavu) glasi:

$$\vec{a}\circ\vec{b}=a_x\cdot b_x+a_y\cdot b_y+a_z\cdot b_z.$$

Računamo:

$$\left.\begin{array}{l}\vec{b}=(1,1,-2)=\vec{i}+\vec{j}-2\cdot\vec{k}\\ \vec{c}=(2,-1,3)=2\cdot\vec{i}-\vec{j}+3\cdot\vec{k}\end{array}\right\}\Rightarrow\vec{c}\circ\vec{b}=c_x\cdot b_x+c_y\cdot b_y+c_z\cdot b_z=2\cdot1+(-1)\cdot1+3\cdot(-2)=2-1-6=-5.$$

Vježba 026

Za vektore $\vec{b}=(2,4,-2)$ i $\vec{c}=(3,4,11)$ nađite $\vec{c}\circ\vec{b}$.

Rezultat: 0.

Zadatak 027 (Ivica, student)

Za vektore $\vec{b}=(1,1,-2)$ i $\vec{c}=(2,-1,3)$ nađite $\vec{b}\times\vec{c}$.

Rješenje 027

Ponovimo!

Formula za vektorski produkt vektora $\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$ i $\vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}$ pomoću njihovih komponenata u pravokutnom koordinatnom sustavu (Kartezijevom koordinatnom sustavu) glasi:

$$\vec{a}\times\vec{b}=\vec{i}\cdot\left(a_y\cdot b_z-a_z\cdot b_y\right)+\vec{j}\cdot\left(a_z\cdot b_x-a_x\cdot b_z\right)+\vec{k}\cdot\left(a_x\cdot b_y-a_y\cdot b_x\right).$$

Računamo:

$$\left.\begin{array}{l}\vec{b}=(1,\,1,\,-2)=\vec{i}+\vec{j}-2\cdot\vec{k}\\ \vec{c}=(2,\,-1,\,3)=2\cdot\vec{i}-\vec{j}+3\cdot\vec{k}\end{array}\right\}\Rightarrow$$

$$\Rightarrow\ \vec{b}\times\vec{c}=\vec{i}\cdot\left(b_y\cdot c_z-b_z\cdot c_y\right)+\vec{j}\cdot\left(b_z\cdot c_x-b_x\cdot c_z\right)+\vec{k}\cdot\left(b_x\cdot c_y-b_y\cdot c_x\right)=$$

$$=\vec{i}\cdot\left(1\cdot 3-(-2)\cdot(-1)\right)+\vec{j}\cdot(-2\cdot 2-1\cdot 3)+\vec{k}\cdot\left(1\cdot(-1)-1\cdot 2\right)=\vec{i}\cdot(3-2)+\vec{j}\cdot(-4-3)+\vec{k}\cdot(-1-2)=$$

$$=\vec{i}-7\cdot\vec{j}-3\cdot\vec{k}.$$

Vježba 027

Za vektore $\vec{b}=(3,\,-1,\,-2)$ i $\vec{c}=(1,\,2,\,-1)$ nađite $\vec{b}\times\vec{c}$.

Rezultat: $5\cdot\vec{i}+\vec{j}+7\cdot\vec{k}$.

Zadatak 028 (Anamarija, Sanela, maturantice gimnazije)

Zadani su vektori $\vec{a}=(3,\,2,\,-5)$, $\vec{b}=(4,\,4,\,-1)$ i $\vec{c}=(6,\,8,\,7)$. Ako postoje realni brojevi α, β i γ takvi da je $\alpha\cdot\vec{a}+\beta\cdot\vec{b}+\gamma\cdot\vec{c}=\vec{0}$, nađite omjer α : β : γ.

Rješenje 028

Ponovimo!

Jednakost vektora $\vec{a}$ i $\vec{b}$:

$$\left.\begin{array}{l}\vec{a}=\left(a_x,\,a_y,\,a_z\right)\\ \vec{b}=\left(b_x,\,b_y,\,b_z\right)\\ \vec{a}=\vec{b}\end{array}\right\}\Rightarrow\left.\begin{array}{l}a_x=b_x\\ a_y=b_y\\ a_z=b_z\end{array}\right\}.$$

$$\alpha\cdot\vec{a}+\beta\cdot\vec{b}+\gamma\cdot\vec{c}=\vec{0}\ \Rightarrow\ \alpha\cdot(3,\,2,\,-5)+\beta\cdot(4,\,4,\,-1)+\gamma\cdot(6,\,8,\,7)=(0,\,0,\,0)\ \Rightarrow$$

$$\Rightarrow\ (3\cdot\alpha,\,2\cdot\alpha,\,-5\cdot\alpha)+(4\cdot\beta,\,4\cdot\beta,\,-1\cdot\beta)+(6\cdot\gamma,\,8\cdot\gamma,\,7\cdot\gamma)=(0,\,0,\,0)\ \Rightarrow$$

$$\Rightarrow\ (3\cdot\alpha+4\cdot\beta+6\cdot\gamma,\,2\cdot\alpha+4\cdot\beta+8\cdot\gamma,\,-5\cdot\alpha-1\cdot\beta+7\cdot\gamma)=(0,\,0,\,0)\ \Rightarrow\left.\begin{array}{l}3\cdot\alpha+4\cdot\beta+6\cdot\gamma=0\\ 2\cdot\alpha+4\cdot\beta+8\cdot\gamma=0\\ -5\cdot\alpha-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}3\cdot\alpha+4\cdot\beta+6\cdot\gamma=0\\ 2\cdot\alpha+4\cdot\beta+8\cdot\gamma=0\ /:2\\ -5\cdot\alpha-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot\alpha+4\cdot\beta+6\cdot\gamma=0\\ \alpha+2\cdot\beta+4\cdot\gamma=0\\ -5\cdot\alpha-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{iz druge jednadžbe}\\ \text{izračunamo α i uvrstimo}\\ \text{u prvu i treću jednadžbu}\end{array}\right]\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}3\cdot\alpha+4\cdot\beta+6\cdot\gamma=0\\ \alpha=-2\cdot\beta-4\cdot\gamma\\ -5\cdot\alpha-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot(-2\cdot\beta-4\cdot\gamma)+4\cdot\beta+6\cdot\gamma=0\\ -5\cdot(-2\cdot\beta-4\cdot\gamma)-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}-6\cdot\beta-12\cdot\gamma+4\cdot\beta+6\cdot\gamma=0\\ 10\cdot\beta+20\cdot\gamma-1\cdot\beta+7\cdot\gamma=0\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}-2\cdot\beta-6\cdot\gamma=0\\ 9\cdot\beta+27\cdot\gamma=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}-2\cdot\beta-6\cdot\gamma=0\ /:(-2)\\ 9\cdot\beta+27\cdot\gamma=0\ /:9\end{array}\right\}\Rightarrow\left.\begin{array}{l}\beta+3\cdot\gamma=0\\ \beta+3\cdot\gamma=0\end{array}\right\}\Rightarrow\ \beta+3\cdot\gamma=0\ \Rightarrow\ \beta=-3\cdot\gamma.$$

Izrazimo α kao funkciju od γ:

$$\left.\begin{array}{l}\beta=-3\cdot\gamma\\ \alpha=-2\cdot\beta-4\cdot\gamma\end{array}\right\}\Rightarrow\ \alpha=-2\cdot(-3\cdot\gamma)-4\cdot\gamma\ \Rightarrow\ \alpha=6\cdot\gamma-4\cdot\gamma\ \Rightarrow\ \alpha=2\cdot\gamma.$$

Gledamo omjer:

$$\left.\begin{array}{l}\alpha=2\cdot\gamma\\ \beta=-3\cdot\gamma\\ \gamma=\gamma\end{array}\right\}\Rightarrow \alpha:\beta:\gamma=2\cdot\gamma:(-3\cdot\gamma):\gamma \Rightarrow \left[\begin{array}{l}\text{članove omjera}\\ \text{dijelimo sa }\gamma\end{array}\right]\Rightarrow \alpha:\beta:\gamma=2:(-3):1.$$

Vježba 028

Zadani su vektori $\vec{a}=(3,2,-5)$, $\vec{b}=(4,4,-1)$ i $\vec{c}=(6,8,7)$. Ako postoje realni brojevi α, β i γ takvi da je $\alpha\cdot\vec{a}+\beta\cdot\vec{b}+\gamma\cdot\vec{c}=\vec{0}$, nađite omjer β : γ : α.

Rezultat: $\beta:\gamma:\alpha=-3:1:2.$

Zadatak 029 (Martina, srednjoškolka)

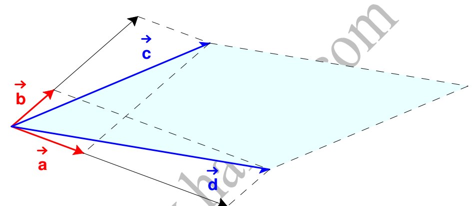
Izračunaj površinu četverokuta razapetog vektorima $\vec{c}=\vec{a}+3\cdot\vec{b}$ i $\vec{d}=3\cdot\vec{a}+\vec{b}$ ako je $\left|\vec{a}\right|=\left|\vec{b}\right|=\sqrt{2}$ i kut između vektora $\vec{a}$ i $\vec{b}$ je $\frac{2\cdot\pi}{3}$.

Rješenje 029

Ponovimo!

$$\vec{x}\times\vec{x}=\vec{0}\quad,\quad \vec{x}\times\vec{y}=-\vec{y}\times\vec{x}\quad,\quad \left|\vec{x}\times\vec{y}\right|=\left|\vec{x}\right|\cdot\left|\vec{y}\right|\cdot\sin\varphi,\ \varphi \text{ je kut između vektora } \vec{x} \text{ i } \vec{y}.$$

$$\sin(\pi-\varphi)=\sin\varphi.$$



Površina četverokuta (paralelograma) razapetog vektorima $\vec{c}$ i $\vec{d}$ računa se po formuli

$$P=\left|\vec{c}\times\vec{d}\right|.$$

Najprije izračunamo vektorski produkt $\vec{c}\times\vec{d}$:

$$\vec{c}\times\vec{d}=\left(\vec{a}+3\cdot\vec{b}\right)\times\left(3\cdot\vec{a}+\vec{b}\right)=3\cdot\vec{a}\times\vec{a}+\vec{a}\times\vec{b}+9\cdot\vec{b}\times\vec{a}+3\cdot\vec{b}\times\vec{b}=3\cdot\vec{0}-\vec{b}\times\vec{a}+9\cdot\vec{b}\times\vec{a}+3\cdot\vec{0}=$$

$$=8\cdot\vec{b}\times\vec{a}.$$

Površina četverokuta iznosi:

$$P=\left|\vec{c}\times\vec{d}\right|\Rightarrow P=\left|8\cdot\vec{b}\times\vec{a}\right|\Rightarrow P=8\cdot\left|\vec{b}\times\vec{a}\right|\Rightarrow P=8\cdot\left|\vec{b}\right|\cdot\left|\vec{a}\right|\cdot\sin\frac{2\cdot\pi}{3}\Rightarrow$$

$$\Rightarrow P=8\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sin\left(\pi-\frac{\pi}{3}\right)\Rightarrow P=8\cdot 2\cdot\sin\frac{\pi}{3}\Rightarrow P=16\cdot\frac{\sqrt{3}}{2}\Rightarrow P=8\cdot\sqrt{3}.$$

Vježba 029

Izračunaj površinu četverokuta razapetog vektorima $\vec{a}=\vec{p}+2\cdot\vec{q}$ i $\vec{b}=2\cdot\vec{p}+\vec{q}$ ako je $\left|\vec{p}\right|=\left|\vec{q}\right|=1$ i kut između vektora $\vec{p}$ i $\vec{q}$ je $\frac{\pi}{3}$.

Rezultat: $\frac{3\cdot\sqrt{3}}{2}.$

Zadatak 030 (Martina, srednjoškolka)

Nađi visinu iz točke P tetraedra čiji su vrhovi P(2, – 1, 0), A(1, – 1, 1), B(2, 0, 2) i C(– 1, 3, 1).

Rješenje 030

Ponovimo!

Ako su zadane točke $A(x_1, y_1, z_1)$ i $B(x_2, y_2, z_2)$, onda su koordinate vektora koji ih spajaju sljedeće:

$$\overrightarrow{AB}=\left(x_2-x_1\right)\cdot\vec{i}+\left(y_2-y_1\right)\cdot\vec{j}+\left(z_2-z_1\right)\cdot\vec{k} \quad , \quad \overrightarrow{BA}=\left(x_1-x_2\right)\cdot\vec{i}+\left(y_1-y_2\right)\cdot\vec{j}+\left(z_1-z_2\right)\cdot\vec{k}.$$

Modul ili apsolutna vrijednost vektora ili duljina vektora

$$\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k} \text{ definira se } \left|\vec{a}\right|=\sqrt{a_x^2+a_y^2+a_z^2}.$$

Vektorski produkt vektora $\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}$ i $\vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}$ je:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \quad \text{ili} \quad \vec{a}\times\vec{b}=\left(a_y\cdot b_z-a_z\cdot b_y\right)\cdot\vec{i}+\left(a_z\cdot b_x-a_x\cdot b_z\right)\cdot\vec{j}+\left(a_x\cdot b_y-a_y\cdot b_x\right)\cdot\vec{k}.$$

Površina paralelograma razapetog vektorima $\vec{a}$ i $\vec{b}$ je:

$$P=\left|\vec{a}\times\vec{b}\right|.$$

Obujam tetraedra (piramide) razapetog vektorima

$$\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k},\ \vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k} \text{ i } \vec{c}=c_x\cdot\vec{i}+c_y\cdot\vec{j}+c_z\cdot\vec{k} \text{ dan je formulama:}$$

$$V=\frac{1}{6}\cdot\left|\left(\vec{a}\times\vec{b}\right)\circ\vec{c}\right| \quad \text{ili} \quad V=\frac{1}{6}\cdot\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}.$$

Obujam tetraedra (piramide) glasi:

$$V=\frac{1}{3}\cdot B\cdot v,$$

gdje je B baza tetraedra, a v njegova visina.

Obujam tetraedra (piramide) ABCP dobije se po formuli

$$V=\frac{1}{6}\cdot\left|\left(\vec{a}\times\vec{b}\right)\circ\vec{c}\right|,$$

gdje su $\vec{a}=\overrightarrow{PA},\ \vec{b}=\overrightarrow{PB},\ \vec{c}=\overrightarrow{PC}$ vektori koji razapinju neki paralelopiped. Dakle je:

$$\left.\begin{array}{l}P(2,-1,0)\ ,\ A(1,-1,1)\\ \vec{a}=\overrightarrow{PA}\end{array}\right\}\Rightarrow\ \vec{a}=(1-2)\cdot\vec{i}+(-1+1)\cdot\vec{j}+(1-0)\cdot\vec{k}\ \Rightarrow\ \vec{a}=-1\cdot\vec{i}+1\cdot\vec{k},$$

$$\left.\begin{array}{l}P(2,-1,0)\ ,\ B(2,0,2)\\ \vec{b}=\overrightarrow{PB}\end{array}\right\}\Rightarrow\ \vec{b}=(2-2)\cdot\vec{i}+(0+1)\cdot\vec{j}+(2-0)\cdot\vec{k}\ \Rightarrow\ \vec{b}=1\cdot\vec{j}+2\cdot\vec{k},$$

$$\left.\begin{array}{l}P(2,-1,0)\ ,\ C(-1,3,1)\\ \vec{c}=\overrightarrow{PC}\end{array}\right\}\Rightarrow\ \vec{c}=(-1-2)\cdot\vec{i}+(3+1)\cdot\vec{j}+(1-0)\cdot\vec{k}\ \Rightarrow\ \vec{c}=-3\cdot\vec{i}+4\cdot\vec{j}+1\cdot\vec{k}.$$

Računamo obujam tetraedra ABCP:

$$\left.\begin{array}{l}\vec{a}=a_x\cdot\vec{i}+a_y\cdot\vec{j}+a_z\cdot\vec{k}\ ,\ \vec{a}=-1\cdot\vec{i}+0\cdot\vec{j}+1\cdot\vec{k}\\ \vec{b}=b_x\cdot\vec{i}+b_y\cdot\vec{j}+b_z\cdot\vec{k}\ ,\ \vec{b}=0\cdot\vec{i}+1\cdot\vec{j}+2\cdot\vec{k}\\ \vec{c}=c_x\cdot\vec{i}+c_y\cdot\vec{j}+c_z\cdot\vec{k}\ ,\ \vec{c}=-3\cdot\vec{i}+4\cdot\vec{j}+1\cdot\vec{k}\end{array}\right\}\Rightarrow V=\frac{1}{6}\cdot\begin{vmatrix}-1&0&1\\0&1&2\\-3&4&1\end{vmatrix}\Rightarrow$$

$$\Rightarrow\left[\begin{array}{l}\text{pomoću Sarrusova pravila izračunamo determinantu}\\ \text{trećeg reda, pogledaj Matrični račun}(40)\text{ Zadatak 005}\end{array}\right]\Rightarrow$$

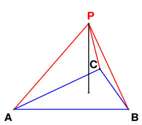
$$\Rightarrow V=\frac{1}{6}\cdot\left|\begin{array}{ccc|cc}-1&0&1&-1&0\\0&1&2&0&1\\-3&4&1&-3&4\end{array}\right|\Rightarrow V=\frac{1}{6}\cdot\left[\left(-1\cdot1\cdot1+0\cdot2\cdot(-3)+1\cdot0\cdot4\right)-\left(-3\cdot1\cdot1+4\cdot2\cdot(-1)+1\cdot0\cdot0\right)\right]\Rightarrow$$

$$\Rightarrow V=\frac{1}{6}\cdot\left[(-1+0+0)-(-3-8+0)\right]\Rightarrow V=\frac{1}{6}\cdot[-1+11]\Rightarrow V=\frac{1}{6}\cdot10\Rightarrow V=\frac{5}{3}.$$

Budući da računamo visinu v iz vrha P tetraedra ABCP, njegova baza je trokut ABC čija je površina jednaka polovici površine paralelograma razapetog vektorima $\vec{a}=\vec{AB}$ i $\vec{b}=\vec{AC}$. Površina trokuta ABC iznosi:

$$\left.\begin{array}{l}A(1,-1,1)\ ,\ B(2,0,2)\\ \vec{a}=\vec{AB}\end{array}\right\}\Rightarrow\vec{a}=(2-1)\cdot\vec{i}+(0+1)\cdot\vec{j}+(2-1)\cdot\vec{k}\Rightarrow\vec{a}=\vec{i}+\vec{j}+\vec{k},$$

$$\left.\begin{array}{l}A(1,-1,1)\ ,\ C(-1,3,1)\\ \vec{b}=\vec{AC}\end{array}\right\}\Rightarrow\vec{b}=(-1-1)\cdot\vec{i}+(3+1)\cdot\vec{j}+(1-1)\cdot\vec{k}\Rightarrow\vec{b}=-2\cdot\vec{i}+4\cdot\vec{j}.$$



$$\left.\begin{array}{l}\vec{a}=\vec{i}+\vec{j}+\vec{k}\\ \vec{b}=-2\cdot\vec{i}+4\cdot\vec{j}+0\cdot\vec{k}\end{array}\right\}\Rightarrow P_{ABC}=\frac{1}{2}\cdot\left|\vec{a}\times\vec{b}\right|\Rightarrow$$

$$\Rightarrow P_{ABC}=\frac{1}{2}\cdot\left|(1\cdot0-1\cdot4)\cdot\vec{i}+(1\cdot(-2)-0\cdot1)\cdot\vec{j}+(1\cdot4-1\cdot(-2))\cdot\vec{k}\right|\Rightarrow P_{ABC}=\frac{1}{2}\cdot\left|-4\cdot\vec{i}-2\cdot\vec{j}+6\cdot\vec{k}\right|\Rightarrow$$

$$\Rightarrow P_{ABC}=\frac{1}{2}\cdot\sqrt{(-4)^2+(-2)^2+6^2}\Rightarrow P_{ABC}=\frac{1}{2}\cdot\sqrt{16+4+36}\Rightarrow P_{ABC}=\frac{1}{2}\cdot\sqrt{56}\Rightarrow$$

$$\Rightarrow\left[\begin{array}{l}\text{djelomično}\\ \text{korjenovanje}\end{array}\right]\Rightarrow P_{ABC}=\frac{1}{2}\cdot\sqrt{4\cdot14}\Rightarrow P_{ABC}=\frac{1}{2}\cdot2\cdot\sqrt{14}\Rightarrow P_{ABC}=\sqrt{14}.$$

Visina v tetraedra ABCP iz točke P iznosi:

$$\left.\begin{array}{l}B=P_{ABC}=\sqrt{14}\\ V=\frac{5}{3}\end{array}\right\}\Rightarrow V=\frac{1}{3}\cdot B\cdot v\Rightarrow v=\frac{3\cdot V}{B}\Rightarrow v=\frac{3\cdot\frac{5}{3}}{\sqrt{14}}\Rightarrow v=\frac{5}{\sqrt{14}}\Rightarrow\left[\begin{array}{l}\text{racionalizacija}\\ \text{nazivnika}\end{array}\right]\Rightarrow$$

$$\Rightarrow v=\frac{5}{\sqrt{14}}\cdot\frac{\sqrt{14}}{\sqrt{14}}\Rightarrow v=\frac{5\cdot\sqrt{14}}{14}.$$

Vježba 030

Nađi visinu iz točke P tetraedra čiji su vrhovi P(0, 0, 7), A(0, – 3, – 1), B(– 2, 0, – 1) i C(– 2, – 3, 5).

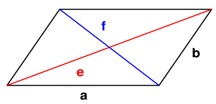
Rezultat: $\sqrt{14}.$

Zadatak 031 (Viki, gimnazija)

Ako za vektore $\vec{a}$ i $\vec{b}$ vrijedi $\left|\vec{a}\right|=6,\ \left|\vec{b}\right|=8,\ \left|\vec{a}-\vec{b}\right|=5,$ koliko je $\left|\vec{a}+\vec{b}\right|$?

Rješenje 031

Ponovimo!



Za stranice a, b i dijagonale e, f paralelograma vrijedi:

$$e^2 + f^2 = 2 \cdot a^2 + 2 \cdot b^2.$$

Budući da se radi o stranicama i dijagonalama paralelograma, slijedi:

$$\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 2 \cdot \left|\vec{a}\right|^2 + 2 \cdot \left|\vec{b}\right|^2 \Rightarrow \left|\vec{a}+\vec{b}\right|^2 = 2 \cdot \left|\vec{a}\right|^2 + 2 \cdot \left|\vec{b}\right|^2 - \left|\vec{a}-\vec{b}\right|^2 \;/\sqrt{} \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right| = \sqrt{2 \cdot \left|\vec{a}\right|^2 + 2 \cdot \left|\vec{b}\right|^2 - \left|\vec{a}-\vec{b}\right|^2} \Rightarrow \left|\vec{a}+\vec{b}\right| = \sqrt{2 \cdot 6^2 + 2 \cdot 8^2 - 5^2} \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right| = \sqrt{72+128-25} \Rightarrow \left|\vec{a}+\vec{b}\right| = \sqrt{175} \Rightarrow \left[\begin{matrix}\text{djelomično} \\ \text{korjenovanje}\end{matrix}\right] \Rightarrow$$

$$\Rightarrow \left|\vec{a}+\vec{b}\right| = \sqrt{25 \cdot 7} \Rightarrow \left|\vec{a}+\vec{b}\right| = 5 \cdot \sqrt{7}.$$

Vježba 031

Ako za vektore $\vec{a}$ i $\vec{b}$ vrijedi $\left|\vec{a}\right| = 6$, $\left|\vec{b}\right| = 8$, $\left|\vec{a}+\vec{b}\right| = 5 \cdot \sqrt{7}$, koliko je $\left|\vec{a}-\vec{b}\right|$?

Rezultat: 5.

Zadatak 032 (Alex99, student)

Koliki je rad sile $\vec{F} = 2 \cdot \vec{i} + 2 \cdot \vec{j} - \vec{k}$ na putu od točke A(1, 1, 2) do točke B(2, 3, 4).

Rješenje 032

Ponovimo!

Ako su dane točke A(x_1, y_1, z_1) i B(x_2, y_2, z_2), onda su koordinate vektora koji ih spaja:

$$\overrightarrow{AB} = \left(x_2 - x_1\right) \cdot \vec{i} + \left(y_2 - y_1\right) \cdot \vec{j} + \left(z_2 - z_1\right) \cdot \vec{k}.$$

Rad stalne sile $\vec{F}$ na putu $\vec{s}$ jednak je skalarnom produktu:

$$W = \vec{F} \circ \vec{s}.$$

Ako su vektori $\vec{F}$ i $\vec{s}$ zadani u koordinatnom sustavu rad stalne sile $\vec{F}$ na putu $\vec{s}$ je:

$$\left.\begin{array}{l} \vec{F} = F_x \cdot \vec{i} + F_y \cdot \vec{j} + F_z \cdot \vec{k} \\ \vec{s} = s_x \cdot \vec{i} + s_y \cdot \vec{j} + s_z \cdot \vec{k} \\ W = \vec{F} \circ \vec{s} \end{array}\right\} \Rightarrow W = F_x \cdot s_x + F_y \cdot s_y + F_z \cdot s_z.$$

Najprije odredimo vektor puta:

$$\vec{s} = \overrightarrow{AB}.$$

$$\left.\begin{array}{l} A\left(x_1, y_1, z_1\right) = A(1, 1, 2) \\ B\left(x_2, y_2, z_2\right) = B(2, 3, 4) \\ \overrightarrow{AB} = \left(x_2 - x_1\right) \cdot \vec{i} + \left(y_2 - y_1\right) \cdot \vec{j} + \left(z_2 - z_1\right) \cdot \vec{k} \end{array}\right\} \Rightarrow \vec{s} = (2-1) \cdot \vec{i} + (3-1) \cdot \vec{j} + (4-2) \cdot \vec{k} \Rightarrow$$

$$\Rightarrow \vec{s} = \vec{i} + 2 \cdot \vec{j} + 2 \cdot \vec{k}.$$

Rad sile iznosi:

$$\left.\begin{array}{l}\vec{F}=2\cdot\vec{i}+2\cdot\vec{j}-\vec{k}\\ \vec{s}=\vec{i}+2\cdot\vec{j}+2\cdot\vec{k}\\ W=F_x\cdot s_x+F_y\cdot s_y+F_z\cdot s_z\end{array}\right\}\Rightarrow W=2\cdot 1+2\cdot 2+(-1)\cdot 2=2+4-2=4.$$

Vježba 032

Koliki je rad sile $\vec{F}=2\cdot\vec{i}+2\cdot\vec{j}-\vec{k}$ na putu od točke $A(2, 2, 3)$ do točke $B(3, 4, 5)$.

Rezultat: 4.

Zadatak 033 (Marina, gimnazija)

Dane su točke A(– 2, 3), B(3, 1), C(– 1, – 1), D(0, 3). Prikažite vektor $\vec{AC}$ kao linearnu kombinaciju vektora $\vec{AB}$ i $\vec{AD}$.

Rješenje 033

Ponovimo!

Ako su $\vec{a}$ i $\vec{b}$ dva vektora i α, β dva realna broja, onda se izraz $\alpha\cdot\vec{a}+\beta\cdot\vec{b}$ zove linearna kombinacija vektora $\vec{a}$ i $\vec{b}$.

Ako su dane točke $A(x_1, y_1)$ i $B(x_2, y_2)$, onda su koordinate vektora koji ih spaja:

$$\vec{AB}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}.$$

Ako je $a\cdot\vec{i}+b\cdot\vec{j}=c\cdot\vec{i}+d\cdot\vec{j}$, onda je $a = c$ i $b = d$.

Budući da je duljina vektora $\vec{AB}$ jednaka duljini dužine $\overline{AB}$, vrijedi:

$$\left|\vec{AB}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$

gdje su $A(x_1, y_1)$, $B(x_2, y_2)$.

Prvo odredimo vektore $\vec{AB}$, $\vec{AC}$ i $\vec{AD}$:

- $$\left.\begin{array}{l}A(x_1, y_1)=A(-2, 3)\\ B(x_2, y_2)=B(3, 1)\\ \vec{AB}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\end{array}\right\}\Rightarrow \vec{AB}=(3-(-2))\cdot\vec{i}+(1-3)\cdot\vec{j}\Rightarrow$$

$$\Rightarrow \vec{AB}=(3+2)\cdot\vec{i}+(1-3)\cdot\vec{j}\Rightarrow \vec{AB}=5\cdot\vec{i}-2\cdot\vec{j}.$$

- $$\left.\begin{array}{l}A(x_1, y_1)=A(-2, 3)\\ C(x_2, y_2)=C(-1, -1)\\ \vec{AC}=(x_2-x_1)\cdot\vec{i}+(y_2-y_1)\cdot\vec{j}\end{array}\right\}\Rightarrow \vec{AC}=(-1-(-2))\cdot\vec{i}+(-1-3)\cdot\vec{j}\Rightarrow$$

$$\Rightarrow \vec{AC}=(-1+2)\cdot\vec{i}+(-1-3)\cdot\vec{j}\Rightarrow \vec{AC}=\vec{i}-4\cdot\vec{j}.$$

$$\bullet \quad \left.\begin{array}{l} A(x_1, y_1) = A(-2, 3) \\ D(x_2, y_2) = D(0, 3) \\ \vec{AD} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{AD} = (0-(-2))\cdot \vec{i} + (3-3)\cdot \vec{j} \Rightarrow$$

$$\Rightarrow \vec{AD} = (0+2)\cdot \vec{i} + (3-3)\cdot \vec{j} \Rightarrow \vec{AD} = 2\cdot \vec{i} + 0\cdot \vec{j} \Rightarrow \vec{AD} = 2\cdot \vec{i}.$$

Sada računamo realne brojeve α i β kako bismo vektor $\vec{AC}$ mogli prikazati kao linearnu kombinaciju vektora $\vec{AB}$ i $\vec{AD}$:

$$\vec{AC} = \alpha\cdot \vec{AB} + \beta\cdot \vec{AD} \Rightarrow \vec{i} - 4\cdot \vec{j} = \alpha\cdot\left(5\cdot \vec{i} - 2\cdot \vec{j}\right) + \beta\cdot\left(2\cdot \vec{i}\right) \Rightarrow$$

$$\Rightarrow \vec{i} - 4\cdot \vec{j} = 5\cdot\alpha\cdot \vec{i} - 2\cdot\alpha\cdot \vec{j} + 2\cdot\beta\cdot \vec{i} \Rightarrow \vec{i} - 4\cdot \vec{j} = (5\cdot\alpha + 2\cdot\beta)\cdot \vec{i} - 2\cdot\alpha\cdot \vec{j} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{koristimo jednakost} \\ \text{vektora} \end{bmatrix} \Rightarrow \left.\begin{array}{l} 1 = 5\cdot\alpha + 2\cdot\beta \\ -4 = -2\cdot\alpha \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5\cdot\alpha + 2\cdot\beta = 1 \\ -2\cdot\alpha = -4 \;/:(-2) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5\cdot\alpha + 2\cdot\beta = 1 \\ \alpha = 2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 5\cdot 2 + 2\cdot\beta = 1 \Rightarrow 10 + 2\cdot\beta = 1 \Rightarrow 2\cdot\beta = 1 - 10 \Rightarrow 2\cdot\beta = -9 \;/:(-2) \Rightarrow \beta = -\frac{9}{2} \Rightarrow \left.\begin{array}{l} \alpha = 2 \\ \beta = -\frac{9}{2} \end{array}\right\}.$$

Dakle, vrijedi:

$$\vec{AC} = 2\cdot \vec{AB} - \frac{9}{2}\cdot \vec{AD}.$$

Vježba 033

Dane su točke A(– 2, 3), B(3, 1), C(– 1, – 1), D(0, 3). Prikažite vektor $\vec{BC}$ kao linearnu kombinaciju vektora $\vec{AD}$ i $\vec{CD}$.

Rezultat: $\vec{BC} = -\frac{7}{4}\cdot \vec{AD} - \frac{1}{2}\cdot \vec{CD}.$

Zadatak 034 (Marina, gimnazija)

Zadana je točka B(2, – 4). Odredite ordinatu točke A(4, y) ako je $\left|\vec{AB}\right| = 2\cdot\sqrt{5}.$

Rješenje 034

Ponovimo!

$$(-a-b)^2 = (a+b)^2 \quad , \quad \sqrt{a^2} = |a| \quad , \quad |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}.$$

Ako su dane točke A(x_1, y_1) i B(x_2, y_2), onda su koordinate vektora koji ih spaja:

$$\vec{AB} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j}.$$

Budući da je duljina vektora $\vec{AB}$ jednaka duljini dužine $\overline{AB}$, vrijedi:

$$\left|\vec{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

gdje su A(x_1, y_1), B(x_2, y_2).

Prvo odredimo vektor $\vec{AB}$:

$$\bullet \quad \left.\begin{array}{l} A(x_1, y_1) = A(4, y) \\ B(x_2, y_2) = B(2, -4) \\ \vec{AB} = (x_2 - x_1)\cdot \vec{i} + (y_2 - y_1)\cdot \vec{j} \end{array}\right\} \Rightarrow \vec{AB} = (2-4)\cdot \vec{i} + (-4-y)\cdot \vec{j} \Rightarrow$$

$$\Rightarrow \vec{AB} = -2\cdot \vec{i} + (-4-y)\cdot \vec{j}.$$

Budući da je $\left|\vec{AB}\right| = 2\cdot\sqrt{5}$, slijedi:

$$\left|\vec{AB}\right| = 2\cdot\sqrt{5} \Rightarrow \sqrt{(-2)^2 + (-4-y)^2} = 2\cdot\sqrt{5} \;/^2 \Rightarrow (-2)^2 + (-4-y)^2 = 4\cdot 5 \Rightarrow 4 + (4+y)^2 = 20 \Rightarrow$$

$$\Rightarrow (4+y)^2 = 20-4 \Rightarrow (4+y)^2 = 16 \;/\sqrt{\ } \Rightarrow \sqrt{(4+y)^2} = \sqrt{16} \Rightarrow |4+y| = 4 \Rightarrow \left.\begin{array}{l} 4+y=4 \\ 4+y=-4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} y = 4-4 \\ y = -4-4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_1 = 0 \\ y_2 = -8 \end{array}\right\}.$$

Dakle, postoje dvije ordinate pa tako i dvije točke za koje vrijedi $\left|\vec{AB}\right| = 2\cdot\sqrt{5}.$

To su točke: $A_1(4, -8)$ i $A_2(4, 0)$.

Vježba 034

Zadana je točka A(2, 5). Odredite apscisu točke B(x, – 1) ako je $\left|\vec{AB}\right| = \sqrt{40}.$

Rezultat: $B_1(0, -1)$ i $B_2(4, -1)$.

Zadatak 035 (Marina, gimnazija)

Koliki je modul (iznos) vektora $\vec{a} + 2\cdot\vec{b}$ ako je $\vec{a} = 2\cdot\vec{i} + 9\cdot\vec{j}$, $\vec{b} = 5\cdot\vec{i} - 2\cdot\vec{j}$.

Rješenje 035

Ponovimo!

Ako vektor zapisujemo u obliku $a\cdot\vec{i} + b\cdot\vec{j}$ duljina vektora (modul, iznos) ima formulu

$$\left|a\cdot\vec{i} + b\cdot\vec{j}\right| = \sqrt{a^2 + b^2}.$$

Najprije odredimo zadani vektor:

$$\vec{a} + 2\cdot\vec{b} = 2\cdot\vec{i} + 9\cdot\vec{j} + 2\cdot\left(5\cdot\vec{i} - 2\cdot\vec{j}\right) = 2\cdot\vec{i} + 9\cdot\vec{j} + 10\cdot\vec{i} - 4\cdot\vec{j} = 12\cdot\vec{i} + 5\cdot\vec{j}.$$

Modul vektora iznosi:

$$\left|\vec{a} + 2\cdot\vec{b}\right| = \left|12\cdot\vec{i} + 5\cdot\vec{j}\right| = \sqrt{12^2 + 5^2} = \sqrt{144+25} = \sqrt{169} = 13.$$

Vježba 035

Koliki je modul (iznos) vektora $3\cdot\vec{a} + \vec{b}$ ako je $\vec{a} = 3\cdot\vec{i} + \vec{j}$, $\vec{b} = 3\cdot\vec{i} + 2\cdot\vec{j}$.

Rezultat: 13.

Zadatak 036 (Vedran, tehnička škola)

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ zadovoljavaju uvjet $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Ako je $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 1$, $\left|\vec{c}\right| = 4$ izračunati $\vec{a}\circ\vec{b} + \vec{b}\circ\vec{c} + \vec{c}\circ\vec{a}$.

Rješenje 036

Ponovimo!

$$\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos 0^{0}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot 1=\left|\vec{a}\right|^{2}\quad,\quad \vec{a}\circ\vec{b}=\vec{b}\circ\vec{a}.$$

Budući da je

$$\vec{a}+\vec{b}+\vec{c}=\vec{0},$$

slijedi

$$\left(\vec{a}+\vec{b}+\vec{c}\right)\circ\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{0}\circ\vec{0}\Rightarrow$$

$$\vec{a}\circ\vec{a}+\vec{a}\circ\vec{b}+\vec{a}\circ\vec{c}+\vec{b}\circ\vec{a}+\vec{b}\circ\vec{b}+\vec{b}\circ\vec{c}+\vec{c}\circ\vec{a}+\vec{c}\circ\vec{b}+\vec{c}\circ\vec{c}=0\Rightarrow$$

$$\Rightarrow\left|\vec{a}\right|^{2}+\vec{a}\circ\vec{b}+\vec{a}\circ\vec{c}+\vec{a}\circ\vec{b}+\left|\vec{b}\right|^{2}+\vec{b}\circ\vec{c}+\vec{a}\circ\vec{c}+\vec{b}\circ\vec{c}+\left|\vec{c}\right|^{2}=0\Rightarrow$$

$$\Rightarrow\left|\vec{a}\right|^{2}+\left|\vec{b}\right|^{2}+\left|\vec{c}\right|^{2}+2\cdot\vec{a}\circ\vec{b}+2\cdot\vec{a}\circ\vec{c}+2\cdot\vec{b}\circ\vec{c}=0\Rightarrow$$

$$\Rightarrow 3^{2}+1^{2}+4^{2}+2\cdot\vec{a}\circ\vec{b}+2\cdot\vec{a}\circ\vec{c}+2\cdot\vec{b}\circ\vec{c}=0\Rightarrow 9+1+16+2\cdot\vec{a}\circ\vec{b}+2\cdot\vec{a}\circ\vec{c}+2\cdot\vec{b}\circ\vec{c}=0\Rightarrow$$

$$\Rightarrow 26+2\cdot\vec{a}\circ\vec{b}+2\cdot\vec{a}\circ\vec{c}+2\cdot\vec{b}\circ\vec{c}=0\Rightarrow 2\cdot\vec{a}\circ\vec{b}+2\cdot\vec{a}\circ\vec{c}+2\cdot\vec{b}\circ\vec{c}=-26\;/:(-2)\Rightarrow$$

$$\Rightarrow\vec{a}\circ\vec{b}+\vec{a}\circ\vec{c}+\vec{b}\circ\vec{c}=-13\Rightarrow\vec{a}\circ\vec{b}+\vec{b}\circ\vec{c}+\vec{c}\circ\vec{a}=-13.$$

Vježba 036

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ zadovoljavaju uvjet $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Ako je $\left|\vec{a}\right|=1,\left|\vec{b}\right|=4,\left|\vec{c}\right|=3$ izračunati $\vec{a}\circ\vec{b}+\vec{b}\circ\vec{c}+\vec{c}\circ\vec{a}$.

Rezultat: – 13.

Zadatak 037 (Kety, gimnazija)

Zadano je $\left|\vec{a}\right|=3,\left|\vec{b}\right|=5.$ Za koju vrijednost skalara α su vektori $\vec{a}+\alpha\cdot\vec{b}$ i $\vec{a}-\alpha\cdot\vec{b}$ međusobno okomiti?

Rješenje 037

Ponovimo!

$$\vec{a}\circ\vec{a}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos 0^{0}=\left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot 1=\left|\vec{a}\right|^{2}\quad,\quad\vec{a}\circ\vec{b}=\vec{b}\circ\vec{a}\quad,\quad\vec{a}\perp\vec{b}\Rightarrow\vec{a}\circ\vec{b}=0.$$

Budući da su vektori $\vec{a}+\alpha\cdot\vec{b}$ i $\vec{a}-\alpha\cdot\vec{b}$ međusobno okomiti, njihov skalarni produkt jednak je nuli:

$$\left(\vec{a}+\alpha\cdot\vec{b}\right)\circ\left(\vec{a}-\alpha\cdot\vec{b}\right)=0\Rightarrow\vec{a}\circ\vec{a}-\alpha\cdot\vec{a}\circ\vec{b}+\alpha\cdot\vec{b}\circ\vec{a}-\alpha^{2}\cdot\vec{b}\circ\vec{b}=0\Rightarrow$$

$$\Rightarrow\left|\vec{a}\right|^{2}-\alpha\cdot\vec{a}\circ\vec{b}+\alpha\cdot\vec{a}\circ\vec{b}-\alpha^{2}\cdot\left|\vec{b}\right|^{2}=0\Rightarrow\left|\vec{a}\right|^{2}-\alpha^{2}\cdot\left|\vec{b}\right|^{2}=0\Rightarrow 3^{2}-\alpha^{2}\cdot 5^{2}=0\Rightarrow$$

$$\Rightarrow 9-25\cdot\alpha^{2}=0\Rightarrow -25\cdot\alpha^{2}=-9\;/:(-25)\Rightarrow\alpha^{2}=\frac{9}{25}\;/\sqrt{}\Rightarrow\alpha=\pm\sqrt{\frac{9}{25}}\Rightarrow\alpha_{1,2}=\pm\frac{3}{5}$$

Vježba 037

Zadano je $\left|\vec{a}\right|=2,\left|\vec{b}\right|=3.$ Za koju vrijednost skalara α su vektori $\vec{a}+\alpha\cdot\vec{b}$ i $\vec{a}-\alpha\cdot\vec{b}$ međusobno okomiti?

Rezultat: $\alpha_{1,2} = \pm\frac{2}{3}.$

Zadatak 038 (Sanela, srednja škola)

Zadane su duljine vektora $\vec{a}$ i $\vec{b}$, $\left|\vec{a}\right| = 3, \left|\vec{b}\right| = 4$ i kut među vektorima $\alpha = \frac{\pi}{3}.$

Izračunajte $\left(\vec{a} - \vec{b}\right)^2.$

Rješenje 038

Ponovimo!

$$(x-y)^2 = x^2 - 2\cdot x\cdot y + y^2.$$

Skalarni produkt vektora $\vec{a}$ i $\vec{b}$ je skalar (realan broj) koji označavamo sa $\vec{a}\circ\vec{b}$ i definiramo ovako:

$$\vec{a}\circ\vec{b} = \left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi,$$

gdje je φ kut između vektora $\vec{a}$ i $\vec{b}$.

$$\vec{a}\circ\vec{b} = \vec{b}\circ\vec{a}.$$

$$\vec{a}^{\,2} = \vec{a}\circ\vec{a} = \left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot\cos 0^0 = \left|\vec{a}\right|\cdot\left|\vec{a}\right|\cdot 1 = \left|\vec{a}\right|^2.$$

$$\left(\vec{a}-\vec{b}\right)^2 = \left(\vec{a}-\vec{b}\right)\circ\left(\vec{a}-\vec{b}\right) = \vec{a}\circ\vec{a} - \vec{a}\circ\vec{b} - \vec{b}\circ\vec{a} + \vec{b}\circ\vec{b} = \vec{a}\circ\vec{a} - \vec{a}\circ\vec{b} - \vec{a}\circ\vec{b} + \vec{b}\circ\vec{b} =$$

$$= \left|\vec{a}\right|^2 - 2\cdot\vec{a}\circ\vec{b} + \left|\vec{b}\right|^2 = \left|\vec{a}\right|^2 - 2\cdot\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\alpha + \left|\vec{b}\right|^2 = 3^2 - 2\cdot 3\cdot 4\cdot\cos\frac{\pi}{3} + 4^2 =$$

$$= 9 - \cancel{2}\cdot 3\cdot 4\cdot\frac{1}{\cancel{2}} + 16 = 9 - 12 + 16 = 13.$$

Vježba 038

Zadane su duljine vektora $\vec{a}$ i $\vec{b}$, $\left|\vec{a}\right| = 3, \left|\vec{b}\right| = 4$ i kut među vektorima $\alpha = \frac{\pi}{3}.$

Izračunajte $\left(\vec{a} + \vec{b}\right)^2.$

Rezultat: 37.

Zadatak 039 (Davor, gimnazija)

Za koju vrijednost α i β su vektori $\vec{a} = -2\cdot\vec{i} + 3\cdot\vec{j} + \beta\cdot\vec{k}$ i $\vec{b} = \alpha\cdot\vec{i} - 6\cdot\vec{j} + 2\cdot\vec{k}$ kolinearni?

Rješenje 039

Ponovimo!

Za dva vektora $\vec{a}$ i $\vec{b}$ kažemo da su kolinearni ako postoji neki realni broj λ tako da vrijedi

$$\vec{b} = \lambda\cdot\vec{a}.$$

Za dva vektora $\vec{a} = a_x\cdot\vec{i} + a_y\cdot\vec{j} + a_z\cdot\vec{k}$ i $\vec{b} = b_x\cdot\vec{i} + b_y\cdot\vec{j} + b_z\cdot\vec{k}$ kažemo da su jednaki ako su im odgovarajuće komponente jednake:

$$\vec{a} = \vec{b} \Leftrightarrow a_x = b_x\ ,\ a_y = b_y\ ,\ a_z = b_z.$$

Budući da zadani vektori moraju biti kolinearni, slijedi:

$$\left.\begin{array}{l}\vec{a}=-2\cdot\vec{i}+3\cdot\vec{j}+\beta\cdot\vec{k}\\ \vec{b}=\alpha\cdot\vec{i}-6\cdot\vec{j}+2\cdot\vec{k}\\ \vec{b}=\lambda\cdot\vec{a}\end{array}\right\}\Rightarrow\alpha\cdot\vec{i}-6\cdot\vec{j}+2\cdot\vec{k}=\lambda\cdot\left(-2\cdot\vec{i}+3\cdot\vec{j}+\beta\cdot\vec{k}\right)\Rightarrow$$

$$\Rightarrow\alpha\cdot\vec{i}-6\cdot\vec{j}+2\cdot\vec{k}=-2\cdot\lambda\cdot\vec{i}+3\cdot\lambda\cdot\vec{j}+\beta\cdot\lambda\cdot\vec{k}.$$

Iz definicije jednakosti dva vektora dobije se:

$$\left.\begin{array}{l}\alpha=-2\cdot\lambda\\ -6=3\cdot\lambda\\ 2=\lambda\cdot\beta\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha=-2\cdot\lambda\\ -3\cdot\lambda=6\ /:(-3)\\ -\lambda\cdot\beta=-2\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha=-2\cdot\lambda\\ \lambda=-2\\ -\lambda\cdot\beta=-2\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha=-2\cdot(-2)\\ -(-2)\cdot\beta=-2\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha=4\\ 2\cdot\beta=-2\ /:2\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha=4\\ \beta=-1\end{array}\right\}.$$

Vježba 039

Za koju vrijednost α i β su vektori $\vec{a}=2\cdot\vec{i}+3\cdot\vec{j}+\beta\cdot\vec{k}$ i $\vec{b}=\alpha\cdot\vec{i}+6\cdot\vec{j}+2\cdot\vec{k}$ kolinearni?

Rezultat: $\alpha = 4$, $\beta = 1$.

Zadatak 040 (Goran, tehnička škola)

Zadano je $\left|\vec{a}\right|=10$, $\left|\vec{b}\right|=2$, $\vec{a}\circ\vec{b}=12$. Izračunajte $\left|\vec{a}\times\vec{b}\right|$.

Rješenje 040

Ponovimo!

$$\cos^2 x+\sin^2 x=1.$$

Skalarni produkt vektora $\vec{a}$ i $\vec{b}$ je skalar (realan broj) koji označavamo sa $\vec{a}\circ\vec{b}$ i definiramo ovako:

$$\vec{a}\circ\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi,$$

gdje je φ kut između vektora $\vec{a}$ i $\vec{b}$.

Duljina vektora $\vec{a}\times\vec{b}$ jednaka je površini paralelograma što ga ti vektori određuju, tj.

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\sin\varphi,$$

gdje je φ kut između vektora $\vec{a}$ i $\vec{b}$.

Iz skalarnog produkta vektora $\vec{a}$ i $\vec{b}$ odredi se cos φ:

$$\vec{a}\circ\vec{b}=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\cos\varphi\Rightarrow\cos\varphi=\frac{\vec{a}\circ\vec{b}}{\left|\vec{a}\right|\cdot\left|\vec{b}\right|}\Rightarrow\cos\varphi=\frac{12}{10\cdot 2}\Rightarrow\cos\varphi=\frac{12}{20}\Rightarrow\cos\varphi=\frac{3}{5}.$$

Budući da je

$$\cos^2\varphi+\sin^2\varphi=1,$$

slijedi:

$$\sin^2\varphi=1-\cos^2\varphi\Rightarrow\sin^2\varphi=1-\left(\frac{3}{5}\right)^2\Rightarrow\sin^2\varphi=1-\frac{9}{25}\Rightarrow\sin^2\varphi=\frac{16}{25}\ /\sqrt{\ }\Rightarrow\sin\varphi=\sqrt{\frac{16}{25}}\Rightarrow\sin\varphi=\frac{4}{5}.$$

Sada je:

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\cdot\left|\vec{b}\right|\cdot\sin\varphi\Rightarrow\left|\vec{a}\times\vec{b}\right|=10\cdot 2\cdot\frac{4}{5}\Rightarrow\left|\vec{a}\times\vec{b}\right|=16.$$

Vježba 040

Zadano je $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 10, \vec{a} \circ \vec{b} = 12.$ Izračunajte $\left|\vec{a} \times \vec{b}\right|.$

Rezultat: 16.