Zadatak 081 (3A, TUPŠ)

Napiši jednadžbu elipse ako je: $a + p = 7$, $e = 2$.

Rješenje 081

Ponovimo!

Osna jednadžba elipse

$$b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2 \quad \text{ili} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Linearni ekscentricitet elipse:

$$e^2 = a^2 - b^2 \Rightarrow e = \sqrt{a^2 - b^2}.$$

Tetiva elipse koja prolazi fokusom i okomita je na veliku os naziva se parametar elipse i njezina duljina se označava s $2 \cdot p$ $(p > 0)$:

$$p = \frac{b^2}{a}.$$

Broj p nazivamo poluparametar elipse.

Rješavamo sustav jednadžbi:

$$\left.\begin{aligned} a + p &= 7 \\ e &= 2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a + \frac{b^2}{a} &= 7 \\ \sqrt{a^2 - b^2} &= 2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a + \frac{b^2}{a} &= 7 \;/\cdot a \\ \sqrt{a^2 - b^2} &= 2 \;/^2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a^2 + b^2 &= 7 \cdot a \\ a^2 - b^2 &= 4 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow 2 \cdot a^2 = 7 \cdot a + 4 \Rightarrow 2 \cdot a^2 - 7 \cdot a - 4 = 0 \Rightarrow \left.\begin{aligned} 2 \cdot a^2 - 7 \cdot a - 4 = 0 \\ a = 2 \;,\; b = -7 \;,\; c = -4 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} a = 2 \;,\; b = -7 \;,\; c = -4 \\ a_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \end{aligned}\right\} \Rightarrow a_{1,2} = \frac{7 \pm \sqrt{49 - 4 \cdot 2 \cdot (-4)}}{2 \cdot 2} \Rightarrow a_{1,2} = \frac{7 \pm \sqrt{49 + 32}}{4} \Rightarrow$$

$$\Rightarrow a_{1,2} = \frac{7 \pm \sqrt{81}}{4} \Rightarrow a_{1,2} = \frac{7 \pm 9}{4} \Rightarrow \left.\begin{aligned} a_1 &= \frac{7+9}{4} \\ a_2 &= \frac{7-9}{4} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a_1 &= \frac{16}{4} \\ a_2 &= \frac{-2}{4} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a_1 &= 4 \\ a_2 &= -\frac{1}{2} \;\; \text{nema smisla} \end{aligned}\right\}.$$

Kvadrat velike poluosi a elipse iznosi:

$$a = 4 \;\; => \;\; a^2 = 16.$$

Kvadrat male poluosi b elipse iznosi:

$$\left.\begin{aligned} a^2 &= 16 \\ a^2 - b^2 &= 4 \end{aligned}\right\} \Rightarrow 16 - b^2 = 4 \Rightarrow -b^2 = 4 - 16 \Rightarrow -b^2 = -12 \;/\cdot(-1) \Rightarrow b^2 = 12.$$

Jednadžba elipse glasi:

$$\left.\begin{aligned} a^2 = 16 \;,\; b^2 = 12 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{aligned}\right\} \Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1 \;/\cdot 48 \Rightarrow 3 \cdot x^2 + 4 \cdot y^2 = 48.$$

Vježba 081

Napiši jednadžbu elipse ako je: $a + p = 5$, $e = 2 \cdot \sqrt{3}$.

Rezultat: $x^2 + 4 \cdot y^2 = 16$.

Zadatak 082 (Marina, maturantica)

Nađi promjer kružnice kojoj su normale pravci $2 \cdot x + 3 \cdot y = 4$, $5 \cdot x + 4 \cdot y = 3$, a pravac $x - y = 0$ tangenta.

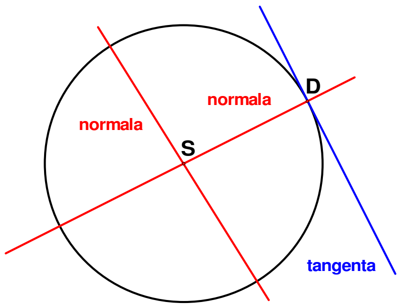
Rješenje 082

Ponovimo!

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Tangenta kružnice je pravac koji s kružnicom ima točno jednu zajedničku točku. Pravac s jednadžbom $y = k \cdot x + l$ dira kružnicu $(x - p)^2 + (y - q)^2 = r^2$ onda i samo onda ako vrijedi

$$r^2 \cdot \left(1 + k^2\right) = (q - k \cdot p - l)^2.$$



Normala na kružnicu je pravac koji prolazi diralištem tangente i okomit je na nju. Jasno je da taj pravac prolazi središtem kružnice.
Budući da normala prolazi središtem kružnice, presjek dviju normala određuje središte kružnice. Koordinate središta dobiju se rješavanjem sustava od dvije linearne jednadžbe sa dvije nepoznanice.

$$\left.\begin{array}{l} 2 \cdot x + 3 \cdot y = 4 \\ 5 \cdot x + 4 \cdot y = 3 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda suprotnih} \\ \text{koeficijenata} \end{array}\right] \Rightarrow \left.\begin{array}{l} 2 \cdot x + 3 \cdot y = 4 \;/ \cdot (-4) \\ 5 \cdot x + 4 \cdot y = 3 \;/ \cdot 3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -8 \cdot x - 12 \cdot y = -16 \\ 15 \cdot x + 12 \cdot y = 9 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 7 \cdot x = -7 \;/ : 7 \Rightarrow x = -1 \Rightarrow \left.\begin{array}{l} x = -1 \\ 2 \cdot x + 3 \cdot y = 4 \end{array}\right\} \Rightarrow 2 \cdot (-1) + 3 \cdot y = 4 \Rightarrow -2 + 3 \cdot y = 4 \Rightarrow$$

$$\Rightarrow 3 \cdot y = 4 + 2 \Rightarrow 3 \cdot y = 6 \;/ : 3 \Rightarrow y = 2.$$

Koordinate središta kružnice su:

$$S(p, q) = S(-1, 2).$$

Pravac $x - y = 0$ napišemo u eksplicitnom obliku kako bismo odredili koeficijent smjera k i odsječak na y osi l.

$$x - y = 0 \Rightarrow -y = -x \;/ \cdot (-1) \Rightarrow y = x \Rightarrow \left\{\begin{array}{l} k = 1 \\ l = 0 \end{array}\right..$$

Iz uvjeta dodira pravca i kružnice izračuna se polumjer r:

$$\left.\begin{array}{l} p = -1 \;,\; q = 2 \;,\; k = 1 \;,\; l = 0 \\ r^2 \cdot \left(1 + k^2\right) = (q - k \cdot p - l)^2 \end{array}\right\} \Rightarrow r^2 \cdot \left(1 + 1^2\right) = (2 - 1 \cdot (-1) - 0)^2 \Rightarrow r^2 \cdot (1 + 1) = (2 + 1)^2 \Rightarrow$$

$$\Rightarrow 2 \cdot r^2 = 3^2 \Rightarrow 2 \cdot r^2 = 9 \;/ : 2 \Rightarrow r^2 = \frac{9}{2} \;/ \sqrt{\;} \Rightarrow r = \sqrt{\frac{9}{2}} \Rightarrow r = \frac{\sqrt{9}}{\sqrt{2}} \Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{l} \text{racionalizacija} \\ \text{nazivnika} \end{array}\right] \Rightarrow r = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow r = \frac{3 \cdot \sqrt{2}}{2}.$$

Promjer kružnice iznosi:

$$d = 2 \cdot r \Rightarrow d = 2 \cdot \frac{3 \cdot \sqrt{2}}{2} \Rightarrow d = 2 \cdot \frac{3 \cdot \sqrt{2}}{2} \Rightarrow d = 3 \cdot \sqrt{2}.$$

Vježba 082

Nađi opseg kružnice kojoj su normale pravci $2 \cdot x + 3 \cdot y = 4$, $5 \cdot x + 4 \cdot y = 3$, a pravac $x - y = 0$ tangenta.

Rezultat: $3 \cdot \sqrt{2} \cdot \pi.$

Zadatak 083 (Matija, gimnazija)

Odredi kut pod kojim se sijeku krivulje $x^2 - y^2 = a^2$ i $x^2 + y^2 = 2 \cdot a^2$, gdje je $a \neq 0$ realan broj.

Rješenje 083

Ponovimo!

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad , \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b} \quad , \quad a^n \cdot a^m = a^{n+m} \quad , \quad |a| = a \text{ za } a \geq 0.$$

Središnja jednadžba kružnice:

$$x^2 + y^2 = r^2.$$

Jednadžba tangente u točki $T(x_1, y_1)$ kružnice:

$$x_1 \cdot x + y_1 \cdot y = r^2.$$

Osna jednadžba hiperbole:

$$b^2 \cdot x^2 - a^2 \cdot y^2 = a^2 \cdot b^2 \quad \text{ili} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Jednadžba tangente u točki $T(x_1, y_1)$ hiperbole:

$$b^2 \cdot x_1 \cdot x - a^2 \cdot y_1 \cdot y = a^2 \cdot b^2.$$

Eksplicitni oblik jednadžbe pravca:

$$y = k \cdot x + l \quad , \quad k = tg\,\alpha \quad , \quad \text{koeficijent smjera}$$

Kut φ između dva pravca koji su određeni jednadžbama:

$$\left.\begin{array}{l} y = k_1 \cdot x + l_1 \\ y = k_2 \cdot x + l_2 \end{array}\right\} \Rightarrow tg\,\varphi = \left|\frac{k_2 - k_1}{1 + k_1 \cdot k_2}\right|.$$

Najprije nađemo presjek hiperbole i kružnice tako što ćemo riješiti sustav od dvije jednadžbe sa dvije nepoznanice.

$$\left.\begin{array}{l} x^2 - y^2 = a^2 \\ x^2 + y^2 = 2 \cdot a^2 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda suprotnih} \\ \text{koeficijenata} \end{array}\right] \Rightarrow 2 \cdot x^2 = 3 \cdot a^2 \;\; /:2 \Rightarrow x^2 = \frac{3 \cdot a^2}{2} \;\; /\sqrt{} \Rightarrow$$

$$\Rightarrow x_{1,2} = \pm\sqrt{\frac{3 \cdot a^2}{2}} \Rightarrow x_{1,2} = \pm a \cdot \sqrt{\frac{3}{2}} \Rightarrow x_{1,2} = \pm\frac{a \cdot \sqrt{3}}{\sqrt{2}} \Rightarrow \left[\begin{array}{l} \text{racionalizacija} \\ \text{nazivnika} \end{array}\right] \Rightarrow$$

$$\Rightarrow x_{1,2} = \pm\frac{a \cdot \sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow x_{1,2} = \pm\frac{a \cdot \sqrt{6}}{2} \Rightarrow \left.\begin{array}{l} x_1 = \frac{a \cdot \sqrt{6}}{2} \\ x_2 = -\frac{a \cdot \sqrt{6}}{2} \end{array}\right\}.$$

Dovoljno je naći jednu točku presjeka jer isti rezultat dobili bismo ako odaberemo bilo koje od 3 preostale presječne točke tih krivulja. Tražimo koordinate jedne točke:

$$\left.\begin{array}{l} x_1 = \frac{a \cdot \sqrt{6}}{2} \\ x^2 + y^2 = 2 \cdot a^2 \end{array}\right\} \Rightarrow \left(\frac{a \cdot \sqrt{6}}{2}\right)^2 + y^2 = 2 \cdot a^2 \Rightarrow \frac{6 \cdot a^2}{4} + y^2 = 2 \cdot a^2 \Rightarrow \frac{3 \cdot a^2}{2} + y^2 = 2 \cdot a^2 \Rightarrow$$

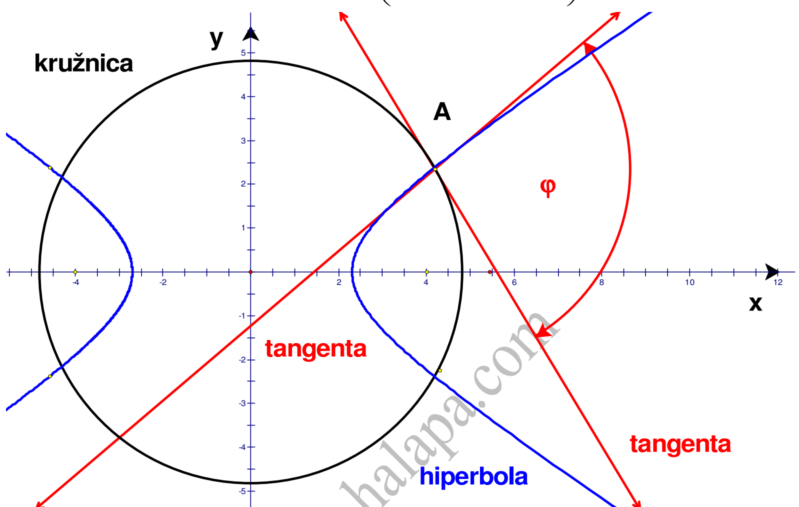
$$\Rightarrow y^2 = 2 \cdot a^2 - \frac{3 \cdot a^2}{2} \Rightarrow y^2 = \frac{4 \cdot a^2 - 3 \cdot a^2}{2} \Rightarrow y^2 = \frac{a^2}{2} \;\; /\sqrt{} \Rightarrow y_{1,2} = \pm\sqrt{\frac{a^2}{2}} \Rightarrow$$

$$\Rightarrow\ y_{1,2}=\pm\frac{\sqrt{a^2}}{\sqrt{2}}\ \Rightarrow\ y_{1,2}=\pm\frac{a}{\sqrt{2}}\ \Rightarrow\ \begin{bmatrix}\text{racionalizacija}\\ \text{nazivnika}\end{bmatrix}\ \Rightarrow\ y_{1,2}=\pm\frac{a}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\ \Rightarrow$$

$$\Rightarrow\ y_{1,2}=\pm\frac{a\cdot\sqrt{2}}{2}\ \Rightarrow\ \left.\begin{matrix} y_1=\dfrac{a\cdot\sqrt{2}}{2}\\ y_2=-\dfrac{a\cdot\sqrt{2}}{2}\end{matrix}\right\}.$$

Jedna od točaka je točka A koja ima koordinate:

$$A\left(x_1,\ y_1\right)=A\left(\frac{a\cdot\sqrt{6}}{2},\ \frac{a\cdot\sqrt{2}}{2}\right).$$



1.inačica

Kut presjeka hiperbole i kružnice jednak je kutu između njihovih tangenata u presječnoj točki A.

- Odredimo tangentu hiperbole u točki A i kut φ_1 koji tangenta zatvara sa pozitivnim dijelom x osi.

$$\left.\begin{matrix} x^2-y^2=a^2\\ \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\end{matrix}\right\}\ \Rightarrow\ \left.\begin{matrix} x^2-y^2=a^2\ \ /:a^2\\ \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\end{matrix}\right\}\ \Rightarrow\ \left.\begin{matrix} \dfrac{x^2}{a^2}-\dfrac{y^2}{a^2}=1\\ \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\end{matrix}\right\}\ \Rightarrow\ \left.\begin{matrix} a^2=a^2\\ b^2=a^2\end{matrix}\right\}.$$

Jednadžba tangente hiperbole u točki A glasi:

$$\left.\begin{matrix} A\left(x_1,\ y_1\right)=A\left(\dfrac{a\cdot\sqrt{6}}{2},\ \dfrac{a\cdot\sqrt{2}}{2}\right)\ ,\ a^2=a^2\ ,\ b^2=a^2\\ b^2\cdot x_1\cdot x-a^2\cdot y_1\cdot y=a^2\cdot b^2\end{matrix}\right\}\ \Rightarrow\ a^2\cdot\frac{a\cdot\sqrt{6}}{2}\cdot x-a^2\cdot\frac{a\cdot\sqrt{2}}{2}\cdot y=a^2\cdot a^2\ \Rightarrow$$

$$\Rightarrow\ \frac{a^3\cdot\sqrt{6}}{2}\cdot x-\frac{a^3\cdot\sqrt{2}}{2}\cdot y=a^4\ \Rightarrow\ \frac{a^3\cdot\sqrt{6}}{2}\cdot x-\frac{a^3\cdot\sqrt{2}}{2}\cdot y=a^4\ \ /\cdot\frac{2}{a^3}\ \Rightarrow$$

$$\Rightarrow\ \sqrt{6}\cdot x-\sqrt{2}\cdot y=2\cdot a\ \Rightarrow\ -\sqrt{2}\cdot y=-\sqrt{6}\cdot x+2\cdot a\ \ /:\left(-\sqrt{2}\right)\ \Rightarrow\ y=\frac{\sqrt{6}}{\sqrt{2}}\cdot x-\frac{2\cdot a}{\sqrt{2}}\ \Rightarrow$$

$$\Rightarrow y=\sqrt{\frac{6}{2}}\cdot x-\frac{2\cdot a}{\sqrt{2}} \Rightarrow y=\sqrt{3}\cdot x-\frac{2\cdot a}{\sqrt{2}} \Rightarrow \left.\begin{matrix} y=\sqrt{3}\cdot x-\frac{2\cdot a}{\sqrt{2}} \\ y=k_1\cdot x+l_1 \end{matrix}\right\} \Rightarrow k_1=\sqrt{3} \Rightarrow \left.\begin{matrix} k_1=\sqrt{3} \\ k_1=tg\,\varphi_1 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow tg\,\varphi_1=\sqrt{3} \Rightarrow \varphi_1=tg^{-1}\left(\sqrt{3}\right) \Rightarrow \varphi_1=\frac{\pi}{3}.$$

- Odredimo tangentu kružnice u točki A i kut φ_2 koji tangenta zatvara sa pozitivnim dijelom x osi.

$$\left.\begin{matrix} x^2+y^2=2\cdot a^2 \\ x^2+y^2=r^2 \end{matrix}\right\} \Rightarrow r^2=2\cdot a^2.$$

Jednadžba tangente kružnice u točki A glasi:

$$\left.\begin{matrix} A\left(x_1,\ y_1\right)=A\left(\frac{a\cdot\sqrt{6}}{2},\ \frac{a\cdot\sqrt{2}}{2}\right)\ ,\ r^2=2\cdot a^2 \\ x_1\cdot x+y_1\cdot y=r^2 \end{matrix}\right\} \Rightarrow \frac{a\cdot\sqrt{6}}{2}\cdot x+\frac{a\cdot\sqrt{2}}{2}\cdot y=2\cdot a^2 \Rightarrow$$

$$\Rightarrow \frac{a\cdot\sqrt{6}}{2}\cdot x+\frac{a\cdot\sqrt{2}}{2}\cdot y=2\cdot a^2 \ /\cdot\frac{2}{a} \Rightarrow \sqrt{6}\cdot x+\sqrt{2}\cdot y=4\cdot a \Rightarrow \sqrt{2}\cdot y=-\sqrt{6}\cdot x+4\cdot a \Rightarrow$$

$$\Rightarrow \sqrt{2}\cdot y=-\sqrt{6}\cdot x+4\cdot a \ /:\sqrt{2} \Rightarrow y=-\frac{\sqrt{6}}{\sqrt{2}}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow y=-\sqrt{\frac{6}{2}}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow$$

$$y=-\sqrt{3}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow \left.\begin{matrix} y=-\sqrt{3}\cdot x+\frac{4\cdot a}{\sqrt{2}} \\ y=k_2\cdot x+l_2 \end{matrix}\right\} \Rightarrow k_2=-\sqrt{3} \Rightarrow \left.\begin{matrix} k_2=-\sqrt{3} \\ k_2=tg\,\varphi_2 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow tg\,\varphi_2=-\sqrt{3} \Rightarrow \varphi_2=tg^{-1}\left(-\sqrt{3}\right) \Rightarrow \varphi_2=\frac{2\cdot\pi}{3}.$$

Kut φ pod kojim se sijeku hiperbola i kružnica jednak je:

$$\varphi=\varphi_2-\varphi_1 \Rightarrow \varphi=\frac{2\cdot\pi}{3}-\frac{\pi}{3} \Rightarrow \varphi=\frac{\pi}{3}.\ \left(\text{manji kut}\right)$$

2.inačica

Kut presjeka hiperbole i kružnice jednak je kutu između njihovih tangenata u presječnoj točki A.

- Odredimo tangentu hiperbole u točki A i koeficijent smjera k_1 tangente.

$$\left.\begin{matrix} x^2-y^2=a^2 \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x^2-y^2=a^2 \ /:a^2 \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \frac{x^2}{a^2}-\frac{y^2}{a^2}=1 \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} a^2=a^2 \\ b^2=a^2 \end{matrix}\right\}.$$

Jednadžba tangente hiperbole u točki A glasi:

$$\left.\begin{matrix} A\left(x_1,\ y_1\right)=A\left(\frac{a\cdot\sqrt{6}}{2},\ \frac{a\cdot\sqrt{2}}{2}\right)\ ,\ a^2=a^2\ ,\ b^2=a^2 \\ b^2\cdot x_1\cdot x-a^2\cdot y_1\cdot y=a^2\cdot b^2 \end{matrix}\right\} \Rightarrow a^2\cdot\frac{a\cdot\sqrt{6}}{2}\cdot x-a^2\cdot\frac{a\cdot\sqrt{2}}{2}\cdot y=a^2\cdot a^2 \Rightarrow$$

$$\Rightarrow \frac{a^3\cdot\sqrt{6}}{2}\cdot x-\frac{a^3\cdot\sqrt{2}}{2}\cdot y=a^4 \Rightarrow \frac{a^3\cdot\sqrt{6}}{2}\cdot x-\frac{a^3\cdot\sqrt{2}}{2}\cdot y=a^4 \;/\cdot\frac{2}{a^3} \Rightarrow$$

$$\Rightarrow \sqrt{6}\cdot x-\sqrt{2}\cdot y=2\cdot a \Rightarrow -\sqrt{2}\cdot y=-\sqrt{6}\cdot x+2\cdot a \;/:\left(-\sqrt{2}\right) \Rightarrow y=\frac{\sqrt{6}}{\sqrt{2}}\cdot x-\frac{2\cdot a}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow y=\sqrt{\frac{6}{2}}\cdot x-\frac{2\cdot a}{\sqrt{2}} \Rightarrow y=\sqrt{3}\cdot x-\frac{2\cdot a}{\sqrt{2}} \Rightarrow \left.\begin{array}{l} y=\sqrt{3}\cdot x-\frac{2\cdot a}{\sqrt{2}} \\ y=k_1\cdot x+l_1 \end{array}\right\} \Rightarrow k_1=\sqrt{3}.$$

- Odredimo tangentu kružnice u točki A i koeficijent smjera k_2 tangente.

$$\left.\begin{array}{l} x^2+y^2=2\cdot a^2 \\ x^2+y^2=r^2 \end{array}\right\} \Rightarrow r^2=2\cdot a^2.$$

Jednadžba tangente kružnice u točki A glasi:

$$\left.\begin{array}{l} A\left(x_1,\, y_1\right)=A\left(\frac{a\cdot\sqrt{6}}{2},\, \frac{a\cdot\sqrt{2}}{2}\right) \;,\; r^2=2\cdot a^2 \\ x_1\cdot x+y_1\cdot y=r^2 \end{array}\right\} \Rightarrow \frac{a\cdot\sqrt{6}}{2}\cdot x+\frac{a\cdot\sqrt{2}}{2}\cdot y=2\cdot a^2 \Rightarrow$$

$$\Rightarrow \frac{a\cdot\sqrt{6}}{2}\cdot x+\frac{a\cdot\sqrt{2}}{2}\cdot y=2\cdot a^2 \;/\cdot\frac{2}{a} \Rightarrow \sqrt{6}\cdot x+\sqrt{2}\cdot y=4\cdot a \Rightarrow \sqrt{2}\cdot y=-\sqrt{6}\cdot x+4\cdot a \Rightarrow$$

$$\Rightarrow \sqrt{2}\cdot y=-\sqrt{6}\cdot x+4\cdot a \;/:\sqrt{2} \Rightarrow y=-\frac{\sqrt{6}}{\sqrt{2}}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow y=-\sqrt{\frac{6}{2}}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow$$

$$y=-\sqrt{3}\cdot x+\frac{4\cdot a}{\sqrt{2}} \Rightarrow \left.\begin{array}{l} y=-\sqrt{3}\cdot x+\frac{4\cdot a}{\sqrt{2}} \\ y=k_2\cdot x+l_2 \end{array}\right\} \Rightarrow k_2=-\sqrt{3}.$$

Kut φ pod kojim se sijeku hiperbola i kružnica jednak je:

$$\left.\begin{array}{l} k_1=\sqrt{3} \;,\; k_2=-\sqrt{3} \\ tg\,\varphi=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right| \end{array}\right\} \Rightarrow tg\,\varphi=\left|\frac{-\sqrt{3}-\sqrt{3}}{1+\sqrt{3}\cdot\left(-\sqrt{3}\right)}\right| \Rightarrow tg\,\varphi=\left|\frac{-2\cdot\sqrt{3}}{1-\left(\sqrt{3}\right)^2}\right| \Rightarrow tg\,\varphi=\left|\frac{-2\cdot\sqrt{3}}{1-3}\right| \Rightarrow$$

$$\Rightarrow tg\,\varphi=\left|\frac{-2\cdot\sqrt{3}}{-2}\right| \Rightarrow tg\,\varphi=\left|\sqrt{3}\right| \Rightarrow tg\,\varphi=\sqrt{3} \Rightarrow \varphi=tg^{-1}\left(\sqrt{3}\right) \Rightarrow \varphi=\frac{\pi}{3}.\;(\text{manji kut})$$

Vježba 083

Odredi veći kut pod kojim se sijeku krivulje $x^2 - y^2 = a^2$ i $x^2 + y^2 = 2\cdot a^2$, gdje je $a \neq 0$ realan broj.

Rezultat: $\frac{2\cdot\pi}{3}.$

Zadatak 084 (Sirena, gimnazija)

Odredi jednadžbe tangenata na elipsu $9\cdot x^2 + 16\cdot y^2 = 144$ koje su paralelne pravcu $x + y - 4 = 0$.

Rješenje 084

Ponovimo!

Osna jednadžba elipse

Elipsa kojoj se središte podudara s ishodištem koordinatnog sustava, smjer glavne osi s x-osi, a smjer sporedne osi s y-osi ima jednadžbu

$$b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2 \quad \text{ili} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

gdje su a i b velika i mala poluos.

Uvjet dodira pravca i elipse

Pravac

$$y = k \cdot x + l$$

dodiruje elipsu

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ako i samo ako vrijedi

$$a^2 \cdot k^2 + b^2 = l^2.$$

Dva su pravca paralelna (usporedna) ako i samo ako su im koeficijenti smjerova jednaki.

$$\left.\begin{array}{l} y = k_1 \cdot x + l_1 \\ y = k_2 \cdot x + l_2 \end{array}\right\} \Rightarrow \begin{bmatrix} \text{uvjet} \\ \text{paralelnosti} \end{bmatrix} \Rightarrow k_1 = k_2.$$

Zadatak rješavamo u dva koraka.

I. Tražena tangenta je paralelna sa zadanim pravcem.

II. Postavimo uvjet dodira traženog pravca (tangente) i elipse.

I. Uvjet paralelnosti	**II.** Uvjet dodira pravca i elipse
$x + y - 4 = 0$ $y = -x + 4.$ Koeficijent smjera zadanog pravca je k = – 1. Budući da su tražene tangente paralelne sa zadanim pravcem, imaju isti koeficijent smjera k = – 1.	$9 \cdot x^2 + 16 \cdot y^2 = 144$ $9 \cdot x^2 + 16 \cdot y^2 = 144 \; /: 144$ $\frac{9 \cdot x^2}{144} + \frac{16 \cdot y^2}{144} = 1$ $\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow a^2 = 16 \; , \; b^2 = 9.$ Iz uvjeta dodira pravca i elipse dobije se l: $\left.\begin{array}{l} a^2 = 16 \; , \; k = -1 \; , \; b^2 = 9 \\ a^2 \cdot k^2 + b^2 = l^2 \end{array}\right\}$ $16 \cdot 1^2 + 9 = l^2$ $16 + 9 = l^2$ $l^2 = 25$ $l^2 = 25 \; / \sqrt{\ }$ $l_{1,2} = \pm \sqrt{25}$ $l_1 = 5 \; , \; l_2 = -5.$

Jednadžbe tangenata su:

$$y = -x + 5 \quad , \quad y = -x - 5.$$

Vježba 084

Odredi jednadžbe tangenata na elipsu $9 \cdot x^2 + 16 \cdot y^2 = 144$ koje su paralelne pravcu $x + y + 4 = 0$.

Rezultat: $y=-x+5 \quad , \quad y=-x+5.$

Zadatak 085 (Ico, gimnazija)

Koliko iznosi najmanja udaljenost elipse $x^2 + 4 \cdot y^2 = 4$ i pravca $x + y - 3 = 0$?

Rješenje 085

Ponovimo!

$$\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y} \quad , \quad |x-y| = y - x \text{ za } x - y < 0.$$

Osna jednadžba elipse

Elipsa kojoj se središte podudara s ishodištem koordinatnog sustava, smjer glavne osi s x-osi, a smjer sporedne osi s y-osi ima jednadžbu

$$b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2 \quad \text{ili} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

gdje su a i b velika i mala poluos.

Uvjet dodira pravca i elipse

Pravac

$$y = k \cdot x + l$$

dodiruje elipsu

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ako i samo ako vrijedi

$$a^2 \cdot k^2 - b^2 = l^2.$$

Koordinate dirališta D tangente i elipse su:

$$D(x_0, y_0) = D\left(-\frac{k \cdot a^2}{l}, \frac{b^2}{l}\right).$$

Dva su pravca paralelna (usporedna) ako i samo ako su im koeficijenti smjerova jednaki.

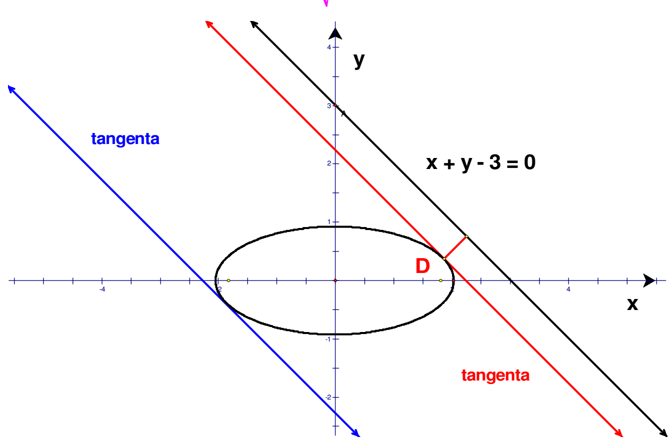
$$\left.\begin{array}{l} y = k_1 \cdot x + l_1 \\ y = k_2 \cdot x + l_2 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{uvjet} \\ \text{paralelnosti} \end{array}\right] \Rightarrow k_1 = k_2.$$

Udaljenost $|T, p|$ točke $T(x_0, y_0)$ i pravca

$$A \cdot x + B \cdot y + C = 0$$

dana je formulom

$$|T, p| = \frac{|A \cdot x_0 + B \cdot y_0 + C|}{\sqrt{A^2 + B^2}}.$$



Zadatak rješavamo u dva koraka.
I. Tražena tangenta je paralelna sa zadanim pravcem.
II. Postavimo uvjet dodira traženog pravca (tangente) i elipse

I. Uvjet paralelnosti	**II.** Uvjet dodira pravca i elipse
$x+y-3=0$ $y=-x+3.$ Koeficijent smjera zadanog pravca je k = – 1. Budući da su tražene tangente paralelne sa zadanim pravcem, imaju isti koeficijent smjera k = – 1.	$x^2+4\cdot y^2=4$ $x^2+4\cdot y^2=4 \;/:4$ $\frac{x^2}{4}+\frac{4\cdot y^2}{4}=1$ $\frac{x^2}{4}+\frac{y^2}{1}=1 \Rightarrow a^2=4 \ , \ b^2=1.$ Iz uvjeta dodira pravca i elipse dobije se l: $\left.\begin{matrix} a^2=4 \ , \ k=-1 \ , \ b^2=1 \\ a^2\cdot k^2+b^2=l^2 \end{matrix}\right\}$ $4\cdot 1^2+1=l^2$ $4+1=l^2$ $l^2=5$ $l^2=5 \;/\sqrt{\ }$ $l_{1,2}=\pm\sqrt{5}$ $l_1=\sqrt{5} \ , \ l_2=-\sqrt{5}.$

Jednadžbe tangenata su:

$$y=-x+\sqrt{5} \quad , \quad y=-x-\sqrt{5}.$$

Tangenta $y=-x+\sqrt{5}$ bliža je pravcu

$$x+y-3=0 \Rightarrow y=-x+3 \text{x + y – 3 = 0}$$

jer joj je odsječak na y-osi pozitivan

$$l=\sqrt{5}.$$

Odredimo koordinate dirališta D(x_0, y_0) tangente $y=-x+\sqrt{5}$ i elipse $\frac{x^2}{4}+\frac{y^2}{1}=1.$

$$\left.\begin{matrix} k=-1 \ , \ l=\sqrt{5} \ , \ a^2=4 \ , \ b^2=1 \\ D\left(x_0, \ y_0\right)=D\left(-\frac{k\cdot a^2}{l}, \ \frac{b^2}{l}\right) \end{matrix}\right\} \Rightarrow D\left(x_0, \ y_0\right)=D\left(-\frac{-1\cdot 4}{\sqrt{5}}, \ \frac{1}{\sqrt{5}}\right) \Rightarrow$$

$$\Rightarrow D\left(x_0, \ y_0\right)=D\left(\frac{4}{\sqrt{5}}, \ \frac{1}{\sqrt{5}}\right).$$

Najmanja udaljenost elipse i pravca x + y – 3 = 0 je, zapravo, udaljenost dirališta D(x_0, y_0) i tog pravca.

$$\left.\begin{array}{l} D\left(x_0,\ y_0\right)=D\left(\dfrac{4}{\sqrt{5}},\ \dfrac{1}{\sqrt{5}}\right) \\ p\ \ldots\ x+y-3=0 \\ A=1\ ,\ B=1\ ,\ C=-3 \end{array}\right\} \Rightarrow \left[\left|D,\ p\right|=\frac{\left|A\cdot x_0+B\cdot y_0+C\right|}{\sqrt{A^2+B^2}}\right] \Rightarrow \left|D,\ p\right|=\frac{\left|1\cdot\dfrac{4}{\sqrt{5}}+1\cdot\dfrac{1}{\sqrt{5}}-3\right|}{\sqrt{1^2+1^2}} \Rightarrow$$

$$\Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{4}{\sqrt{5}}+\dfrac{1}{\sqrt{5}}-3\right|}{\sqrt{1+1}} \Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{5}{\sqrt{5}}-3\right|}{\sqrt{2}} \Rightarrow \left[\begin{array}{l}\text{racionalizacija}\\ \text{razlomka}\end{array}\right] \Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{5}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}}-3\right|}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{5\cdot\sqrt{5}}{\sqrt{25}}-3\right|}{\sqrt{2}} \Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{5\cdot\sqrt{5}}{5}-3\right|}{\sqrt{2}} \Rightarrow \left|D,\ p\right|=\frac{\left|\dfrac{5\cdot\sqrt{5}}{5}-3\right|}{\sqrt{2}} \Rightarrow \left|D,\ p\right|=\frac{\left|\sqrt{5}-3\right|}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{l}\text{racionalizacija}\\ \text{nazivnika}\end{array}\right] \Rightarrow \left|D,\ p\right|=\frac{\left|\sqrt{5}-3\right|}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \Rightarrow \left|D,\ p\right|=\frac{\left|\sqrt{10}-3\cdot\sqrt{2}\right|}{\sqrt{4}} \Rightarrow$$

$$\Rightarrow \left|D,\ p\right|=\frac{\left|\sqrt{10}-3\cdot\sqrt{2}\right|}{2} \Rightarrow \left[\sqrt{10}-3\cdot\sqrt{2}<0\right] \Rightarrow \left|D,\ p\right|=\frac{-\left(\sqrt{10}-3\cdot\sqrt{2}\right)}{2} \Rightarrow$$

$$\Rightarrow \left|D,\ p\right|=\frac{3\cdot\sqrt{2}-\sqrt{10}}{2}.$$

Vježba 085

Koliko iznosi najmanja udaljenost elipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ i pravca $\frac{x}{3}+\frac{y}{3}=1$?

Rezultat: $\frac{3\cdot\sqrt{2}-\sqrt{10}}{2}.$

Zadatak 086 (Antoaneta, maturantica gimnazije)

Odredi zajedničke tangente krivulja $x^2 + 4 \cdot y^2 = 8$, $y^2 = 4 \cdot x$.

Rješenje 086

Ponovimo!

$$a^n\cdot a^m=a^{n+m}.$$

Uvjet dodira pravca i elipse
Pravac

$$y=k\cdot x+l$$

dodiruje elipsu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ako i samo ako vrijedi

$$a^2\cdot k^2+b^2=l^2.$$

Uvjet dodira pravca i parabole
Pravac

$$y=k\cdot x+l$$

dodiruje parabolu

$$y^2=2\cdot p\cdot x$$

ako i samo ako vrijedi

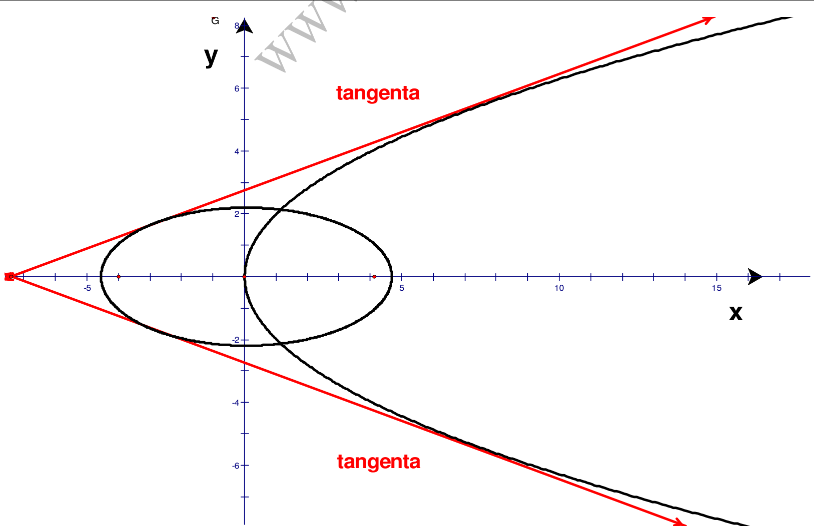
$$p=2\cdot k\cdot l.$$

Budući da je pravac y = k · x + l tangenta elipse i parabole, moramo postaviti dva uvjeta dodira.

I. Uvjet dodira pravca i elipse

II. Uvjet dodira pravca i parabole

I. Uvjet dodira pravca i elipse	**II.** Uvjet dodira pravca i parabole
$x^2+4\cdot y^2=8$ $x^2+4\cdot y^2=8\ /:8$ $\frac{x^2}{8}+\frac{4\cdot y^2}{8}=1$ $\frac{x^2}{8}+\frac{y^2}{2}=1.$ Sada je: $\left.\begin{array}{l}\frac{x^2}{8}+\frac{y^2}{2}=1\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\end{array}\right\}\Rightarrow\left.\begin{array}{l}a^2=8\\ b^2=2\end{array}\right\}.$ Uvjet dodira: $\left.\begin{array}{l}a^2=8\ ,\ b^2=2\\ a^2\cdot k^2+b^2=l^2\end{array}\right\}\Rightarrow 8\cdot k^2+2=l^2.$	$y^2=4\cdot x$ Sada je: $\left.\begin{array}{l}y^2=4\cdot x\\ y^2=2\cdot p\cdot x\end{array}\right\}\Rightarrow 2\cdot p=4\ /:2\ \Rightarrow\ p=2.$ Uvjet dodira: $\left.\begin{array}{l}p=2\\ p=2\cdot k\cdot l\end{array}\right\}\Rightarrow 2=2\cdot k\cdot l\ /:2\ \Rightarrow\ k\cdot l=1.$



Rješavamo sustav jednadžbi:

$$\left.\begin{array}{l}8\cdot k^2+2=l^2\\ k\cdot l=1\end{array}\right\}\Rightarrow\left.\begin{array}{l}8\cdot k^2+2=l^2\\ l=\frac{1}{k}\end{array}\right\}\Rightarrow 8\cdot k^2+2=\left(\frac{1}{k}\right)^2\Rightarrow 8\cdot k^2+2=\frac{1}{k^2}\ /\cdot k^2\ \Rightarrow$$

$$\Rightarrow 8\cdot k^4+2\cdot k^2=1 \Rightarrow 8\cdot k^4+2\cdot k^2-1=0 \Rightarrow \left[\begin{array}{l}\text{bikvadratna jednadžba,}\\ \text{supstitucija } t=k^2\end{array}\right] \Rightarrow 8\cdot t^2+2\cdot t-1=0 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}8\cdot t^2+2\cdot t-1=0\\ a=8\ ,\ b=2\ ,\ c=-1\end{array}\right\} \Rightarrow \left.\begin{array}{l}a=8\ ,\ b=2\ ,\ c=-1\\ t_{1,2}=\dfrac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}\end{array}\right\} \Rightarrow t_{1,2}=\frac{-2\pm\sqrt{4-4\cdot 8\cdot(-1)}}{2\cdot 8} \Rightarrow$$

$$\Rightarrow t_{1,2}=\frac{-2\pm\sqrt{4+32}}{16} \Rightarrow t_{1,2}=\frac{-2\pm\sqrt{36}}{16} \Rightarrow t_{1,2}=\frac{-2\pm 6}{16} \Rightarrow \left.\begin{array}{l}t_1=\dfrac{-2+6}{16}\\ t_2=\dfrac{-2-6}{16}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}t_1=\dfrac{4}{16}\\ t_2=-\dfrac{8}{16}\end{array}\right\} \Rightarrow \left.\begin{array}{l}t_1=\dfrac{1}{4}\\ t_2=-\dfrac{1}{2}\end{array}\right\}.$$

Vraćamo se na supstituciju (zamjenu).

- $$\left.\begin{array}{l}t=\dfrac{1}{4}\\ t=k^2\end{array}\right\} \Rightarrow k^2=\frac{1}{4}\ /\sqrt{\ } \Rightarrow k_{1,2}=\pm\sqrt{\frac{1}{4}} \Rightarrow k_{1,2}=\pm\frac{1}{2} \Rightarrow \left.\begin{array}{l}k_1=\dfrac{1}{2}\\ k_2=-\dfrac{1}{2}\end{array}\right\}.$$

Računamo pripadni l odsječak na osi y:

$$1)\quad \left.\begin{array}{l}k_1=\dfrac{1}{2}\\ l_1=\dfrac{1}{k_1}\end{array}\right\} \Rightarrow l_1=\frac{1}{\frac{1}{2}} \Rightarrow l_1=2.$$

$$2)\quad \left.\begin{array}{l}k_2=-\dfrac{1}{2}\\ l_2=\dfrac{1}{k_2}\end{array}\right\} \Rightarrow l_2=\frac{1}{-\frac{1}{2}} \Rightarrow l_2=-2.$$

- $$\left.\begin{array}{l}t=-\dfrac{1}{2}\\ t=k^2\end{array}\right\} \Rightarrow k^2=-\frac{1}{2}\ /\sqrt{\ } \Rightarrow k_{1,2}=\pm\sqrt{-\frac{1}{2}}.\ \text{nema smisla jer se dobiju imaginarni brojevi}$$

Jednadžbe traženih tangenata glase:

- $$\left.\begin{array}{l}k_1=\dfrac{1}{2}\ ,\ l_1=2\\ y=k_1\cdot x+l_1\end{array}\right\} \Rightarrow y=\frac{1}{2}\cdot x+2.$$
- $$\left.\begin{array}{l}k_2=-\dfrac{1}{2}\ ,\ l_2=-2\\ y=k_2\cdot x+l_2\end{array}\right\} \Rightarrow y=-\frac{1}{2}\cdot x-2.$$

Vježba 086

Napisati jednadžbe zajedničkih tangenata krivulja $y^2 = 8 \cdot x$, $x^2 + y^2 = 2$.

Rezultat: $y = \pm x \pm 2$.

Zadatak 087 (Ivan, maturant gimnazije)

U točkama M(2, y > 0) i N(8, y > 0) parabole $y^2 = 18 \cdot x$ povuku se tangente. Odredi kut pod kojim se sijeku te tangente.

Rješenje 087

Ponovimo!

Tangenta na parabolu

$$y^2 = 2 \cdot p \cdot x$$

u njezinoj točki $D(x_0, y_0)$ ima jednadžbu

$$y \cdot y_0 = p \cdot (x + x_0).$$

Kut između dva pravca

Ako su dva pravca dana eksplicitnim jednadžbama

$$\left.\begin{array}{l} y = k_1 \cdot x + l_1 \\ y = k_2 \cdot x + l_2 \end{array}\right\}$$

i ako nisu okomiti, tada za kut φ između ta dva pravca vrijedi

$$tg\, \varphi = \left| \frac{k_2 - k_1}{1 + k_1 \cdot k_2} \right|.$$

Odredimo ordinate točaka M(2, y > 0) i N(8, y > 0) koje pripadaju paraboli $y^2 = 18 \cdot x$.

Točka M(x, y)

$$\left.\begin{array}{l} M(x,\ y) = M(2,\ y) \\ y^2 = 18 \cdot x \end{array}\right\} \Rightarrow y^2 = 18 \cdot 2 \Rightarrow y^2 = 36 \ /\sqrt{\ } \Rightarrow y_{1,2} = \pm\sqrt{36} \Rightarrow \left.\begin{array}{l} y_1 = 6 \\ y_2 = -6 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{uvjet} \\ y > 0 \end{bmatrix} \Rightarrow y = 6 \Rightarrow M(2,\ 6).$$

Točka N(x, y)

$$\left.\begin{array}{l} N(x,\ y) = N(8,\ y) \\ y^2 = 18 \cdot x \end{array}\right\} \Rightarrow y^2 = 18 \cdot 8 \Rightarrow y^2 = 144 \ /\sqrt{\ } \Rightarrow y_{1,2} = \pm\sqrt{144} \Rightarrow \left.\begin{array}{l} y_1 = 12 \\ y_2 = -12 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{uvjet} \\ y > 0 \end{bmatrix} \Rightarrow y = 12 \Rightarrow N(8,\ 12).$$

Izračunamo parametar p zadane parabole:

$$\left.\begin{array}{l} y^2 = 2 \cdot p \cdot x \\ y^2 = 18 \cdot x \end{array}\right\} \Rightarrow 2 \cdot p = 18 \ /:2 \Rightarrow p = 9.$$

Jednadžbe tangenata u točkama M i N glase:

- $$\left.\begin{array}{l} M(x_0,\ y_0) = M(2,\ 6)\ ,\ p = 9 \\ y \cdot y_0 = p \cdot (x + x_0) \end{array}\right\} \Rightarrow y \cdot 6 = 9 \cdot (x + 2) \Rightarrow 6 \cdot y = 9 \cdot x + 18 \ /:6 \Rightarrow$$

$$\Rightarrow y = \frac{9}{6} \cdot x + 3 \Rightarrow y = \frac{3}{2} \cdot x + 3 \Rightarrow k_1 = \frac{3}{2}.$$

- $$\left.\begin{array}{l} N\left(x_0, y_0\right)=N(8, 12)\ ,\ p=9 \\ y\cdot y_0=p\cdot\left(x+x_0\right)\end{array}\right\}\Rightarrow y\cdot 12=9\cdot(x+8)\Rightarrow 12\cdot y=9\cdot x+72\ /:12\Rightarrow$$

$$\Rightarrow y=\frac{9}{12}\cdot x+6\Rightarrow y=\frac{3}{4}\cdot x+6\Rightarrow k_2=\frac{3}{4}.$$

Kut tangenata φ iznosi:

$$\left.\begin{array}{l} k_1=\frac{3}{2}\ ,\ k_2=\frac{3}{4} \\ tg\,\varphi=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right|\end{array}\right\}\Rightarrow tg\,\varphi=\left|\frac{\frac{3}{4}-\frac{3}{2}}{1+\frac{3}{2}\cdot\frac{3}{4}}\right|\Rightarrow tg\,\varphi=\left|\frac{\frac{3-6}{4}}{1+\frac{9}{8}}\right|\Rightarrow tg\,\varphi=\left|\frac{-\frac{3}{4}}{\frac{17}{8}}\right|\Rightarrow tg\,\varphi=\frac{\frac{3}{4}}{\frac{17}{8}}\Rightarrow$$

$$\Rightarrow tg\,\varphi=\frac{\frac{3}{1}}{\frac{17}{2}}\Rightarrow tg\,\varphi=\frac{6}{17}\Rightarrow\varphi=tg^{-1}\left(\frac{6}{17}\right)\Rightarrow\varphi=19^0 26'24''.$$

Vježba 087

Koliki je kut između tangenata na paraboli $y^2 = 4\cdot x$ povučenih u točkama s apscisom 4?

Rezultat: Naputak: Točke na paraboli s apscisom 4 su M(4, 4) i N(4, 0 – 4). Tangente u njima su $y=\frac{1}{2}\cdot x+2$ i $y=-\frac{1}{2}\cdot x-2$ i kut između njih iznosi φ = 53.13°.

Zadatak 088 (Mario, gimnazija)

Pravac $x - 2\cdot y + 8 = 0$ je zajednička tangenta elipse $b^2\cdot x^2 + a^2\cdot y^2 = a^2\cdot b^2$ i sa njom konfokalne parabole $y^2 = 2\cdot p\cdot x$. Nađi jednadžbu parabole i elipse.

Rješenje 088

Ponovimo!

Krivulje koje imaju iste fokuse (žarišta) nazivamo konfokalnim krivuljama.

Elipsa kojoj se središte podudara s ishodištem koordinatnog sustava, smjer glavne osi s x-osi, a smjer sporedne osi s y-osi ima jednadžbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow b^2\cdot x^2+a^2\cdot y^2=a^2\cdot b^2,$$

gdje su a i b velika i mala poluos.

Uvjet dodira pravca i elipse

Pravac

$$y=k\cdot x+l$$

dodiruje elipsu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ako i samo ako vrijedi

$$a^2\cdot k^2+b^2=l^2.$$

Linearni ekscentricitet e elipse definira se:

$$e^2=a^2-b^2\Rightarrow e=\sqrt{a^2-b^2}.$$

Žarišta (fokusi) elipse imaju koordinate:

$$F_1(-e, 0)\ ,\ F_2(e, 0).$$

Parabola kojoj se os podudara s x-osi koordinatnog sustava, vrh sa ishodištem, a fokus se nalazi na pozitivnom dijelu x-osi ima jednadžbu

$$y^2 = 2 \cdot p \cdot x,$$

pri čemu je p parametar parabole.
Ukoliko se fokus F nalazi na pozitivnom dijelu x-osi, njegove koordinate su:

$$F\left(\frac{p}{2}, 0\right).$$

Uvjet dodira pravca i parabole
Pravac

$$y = k \cdot x + l$$

dodiruje parabolu

$$y^2 = 2 \cdot p \cdot x$$

ako i samo ako vrijedi

$$p = 2 \cdot k \cdot l.$$

Jednadžbu zadanog pravca napišemo u eksplicitnom obliku y = k · x + l da bismo odredili koeficijent smjera k i odsječak na y-osi l:

$$x - 2 \cdot y + 8 = 0 \Rightarrow -2 \cdot y = -x - 8 \;/:(-2) \Rightarrow y = \frac{1}{2} \cdot x + 4 \Rightarrow \left.\begin{matrix} k = \frac{1}{2} \\ l = 4 \end{matrix}\right\}.$$

Iz uvjeta dodira pravca i parabole izračunamo parametar p pa jednadžba parabole glasi:

$$\left.\begin{matrix} k = \frac{1}{2}, \; l = 4 \\ p = 2 \cdot k \cdot l \end{matrix}\right\} \Rightarrow p = 2 \cdot \frac{1}{2} \cdot 4 \Rightarrow p = 4 \Rightarrow \left.\begin{matrix} p = 4 \\ y^2 = 2 \cdot p \cdot x \end{matrix}\right\} \Rightarrow y^2 = 2 \cdot 4 \cdot x \Rightarrow y^2 = 8 \cdot x.$$

Uporabom uvjeta dodira pravca i elipse dobije se jednadžba:

$$\left.\begin{matrix} k = \frac{1}{2}, \; l = 4 \\ a^2 \cdot k^2 + b^2 = l^2 \end{matrix}\right\} \Rightarrow a^2 \cdot \left(\frac{1}{2}\right)^2 + b^2 = 4^2 \Rightarrow \frac{1}{4} \cdot a^2 + b^2 = 16 \;/\cdot 4 \Rightarrow a^2 + 4 \cdot b^2 = 64.$$

Budući da su elipsa i parabola konfokalne, imaju iste fokuse

$$F(e, 0) = F\left(\frac{p}{2}, 0\right) \Rightarrow e = \frac{p}{2} \Rightarrow \sqrt{a^2 - b^2} = \frac{4}{2} \Rightarrow \sqrt{a^2 - b^2} = 2 \;/^2 \Rightarrow a^2 - b^2 = 4.$$

Riješimo sustav jednadžbi:

$$\left.\begin{matrix} a^2 + 4 \cdot b^2 = 64 \\ a^2 - b^2 = 4 \end{matrix}\right\} \Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow \left.\begin{matrix} a^2 + 4 \cdot b^2 = 64 \\ a^2 - b^2 = 4 \;/\cdot(-1) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} a^2 + 4 \cdot b^2 = 64 \\ -a^2 + b^2 = -4 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow 5 \cdot b^2 = 60 \;/:5 \Rightarrow b^2 = 12 \Rightarrow \left.\begin{matrix} b^2 = 12 \\ a^2 - b^2 = 4 \end{matrix}\right\} \Rightarrow a^2 - 12 = 4 \Rightarrow a^2 = 4 + 12 \Rightarrow a^2 = 16.$$

Jednadžba elipse glasi:

$$\left.\begin{matrix} a^2 = 16, \; b^2 = 12 \\ b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2 \end{matrix}\right\} \Rightarrow 12 \cdot x^2 + 16 \cdot y^2 = 16 \cdot 12 \Rightarrow 12 \cdot x^2 + 16 \cdot y^2 = 192.$$

Vježba 088

Pravac $\frac{x}{-8}+\frac{y}{4}=1$ je zajednička tangenta elipse $b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2$ i sa njom konfokalne parabole $y^2 = 2 \cdot p \cdot x$. Nađi jednadžbu parabole i elipse.

Rezultat: $y^2 = 8 \cdot x$, $12 \cdot x^2 + 16 \cdot y^2 = 192$.

Zadatak 089 (Ana, gimnazija)

Kroz točku T(1, 1) unutar elipse $4 \cdot x^2 + 9 \cdot y^2 = 36$ treba postaviti tetivu kojoj je točka T polovište. Nađi jednadžbu pravca na kojem leži ta tetiva.

Rješenje 089

Ponovimo!

$$a^2-b^2=(a-b)\cdot(a+b).$$

Polovište dužine

Koordinate polovišta P dužine $\overline{AB}$, $A(x_1, y_1)$, $B(x_2, y_2)$, su

$$P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Pravac točkama $A(x_1, y_1)$, $B(x_2, y_2)$, $x_1 \neq x_2$, ima koeficijent smjera

$$k=\frac{y_2-y_1}{x_2-x_1}.$$

Jednadžba pravca točkom T s danim koeficijentom smjera k

Pravac kroz točku $T(x_1, y_1)$ s koeficijentom smjera k ima jednadžbu

$$y-y_1=k\cdot(x-x_1).$$

Neka su krajnje točke tražene tetive $A(x_1, y_1)$ i $B(x_2, y_2)$.

Budući da je točka T(1, 1) polovište tražene tetive $\overline{AB}$, slijedi:

$$\left.\begin{array}{l}\frac{x_1+x_2}{2}=1\\ \frac{y_1+y_2}{2}=1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\frac{x_1+x_2}{2}=1 \ /\cdot 2\\ \frac{y_1+y_2}{2}=1 \ /\cdot 2\end{array}\right\} \Rightarrow \left.\begin{array}{l}x_1+x_2=2\\ y_1+y_2=2\end{array}\right\}.$$

Točke A i B su na elipsi pa njihove koordinate uvrstimo u jednadžbu elipse:

$$\left.\begin{array}{l}A(x, y)=A(x_1, y_1)\\ 4\cdot x^2+9\cdot y^2=36\end{array}\right\} \Rightarrow 4\cdot x_1^2+9\cdot y_1^2=36,$$

$$\left.\begin{array}{l}B(x, y)=B(x_2, y_2)\\ 4\cdot x^2+9\cdot y^2=36\end{array}\right\} \Rightarrow 4\cdot x_2^2+9\cdot y_2^2=36.$$

Oduzimanjem prve jednakosti od druge i rastavljanjem na faktore dobije se koeficijent smjera traženog pravca:

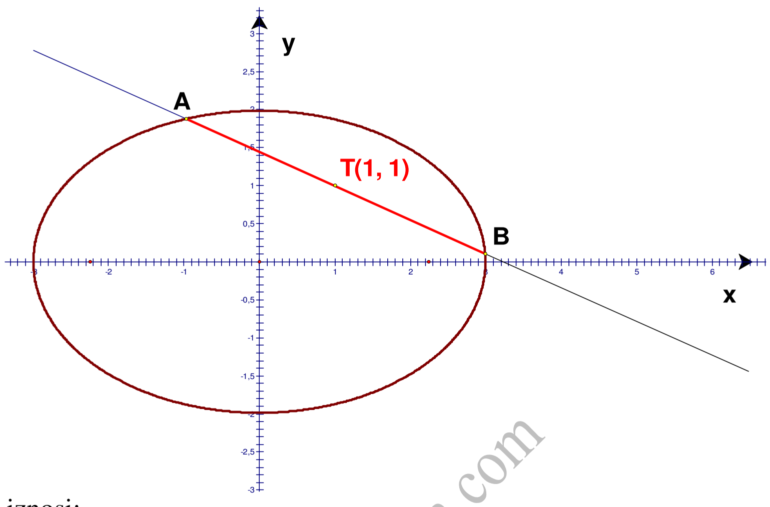
$$\left.\begin{array}{l}4\cdot x_1^2+9\cdot y_1^2=36\\ 4\cdot x_2^2+9\cdot y_2^2=36\end{array}\right\} \Rightarrow \left[\begin{array}{l}\text{oduzmemo}\\ \text{jednakosti}\end{array}\right] \Rightarrow 4\cdot x_2^2+9\cdot y_2^2-4\cdot x_1^2-9\cdot y_1^2=36-36 \Rightarrow$$

$$\Rightarrow\ 4\cdot x_2^2-4\cdot x_1^2+9\cdot y_2^2-9\cdot y_1^2=0\ \Rightarrow\ 4\cdot\left(x_2^2-x_1^2\right)+9\cdot\left(y_2^2-y_1^2\right)=0\ \Rightarrow$$

$$\Rightarrow\ 4\cdot\left(x_2-x_1\right)\cdot\left(x_2+x_1\right)+9\cdot\left(y_2-y_1\right)\cdot\left(y_2+y_1\right)=0\ \Rightarrow\ 4\cdot\left(x_2-x_1\right)\cdot 2+9\cdot\left(y_2-y_1\right)\cdot 2=0\ \Rightarrow$$

$$\Rightarrow\ 4\cdot\left(x_2-x_1\right)\cdot 2+9\cdot\left(y_2-y_1\right)\cdot 2=0\ /:2\ \Rightarrow\ 4\cdot\left(x_2-x_1\right)+9\cdot\left(y_2-y_1\right)=0\ \Rightarrow$$

$$\Rightarrow\ 9\cdot\left(y_2-y_1\right)=-\,4\cdot\left(x_2-x_1\right)\ /\cdot\frac{1}{9\cdot\left(x_2-x_1\right)}\ \Rightarrow\ \frac{y_2-y_1}{x_2-x_1}=-\frac{4}{9}\ \Rightarrow\ k=-\frac{4}{9}.$$



Jednadžba tetive iznosi:

$$\left.\begin{array}{l} T\left(x_1,\ y_1\right)=T\left(1,\ 1\right)\ ,\ k=-\dfrac{4}{9} \\ y-y_1=k\cdot\left(x-x_1\right) \end{array}\right\}\ \Rightarrow\ y-1=-\frac{4}{9}\cdot\left(x-1\right)\ /\cdot 9\ \Rightarrow\ 9\cdot\left(y-1\right)=-4\cdot\left(x-1\right)\ \Rightarrow$$

$$\Rightarrow\ 9\cdot y-9=-4\cdot x+4\ \Rightarrow\ 4\cdot x+9\cdot y-9-4=0\ \Rightarrow\ 4\cdot x+9\cdot y-13=0.$$

Vježba 089

Kroz točku T(1, 1) unutar elipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ treba postaviti tetivu kojoj je točka T polovište. Nađi jednadžbu pravca na kojem leži ta tetiva.

Rezultat: $4\cdot x+9\cdot y-13=0.$

Zadatak 090 (Tomo, srednja škola)

Pravac $2\cdot x+y-12=0$ siječe parabolu $y^2=4\cdot x$. Odredi kut između tangenata parabole u tim točkama.

Rješenje 090

Ponovimo!

$$\left(a-b\right)^2=a^2-2\cdot a\cdot b+b^2.$$

Parabola kojoj se os podudara s x – osi koordinatnog sustava, vrh sa ishodištem, a fokus se nalazi na pozitivnom dijelu x – osi ima jednadžbu

$$y^2=2\cdot p\cdot x,$$

pri čemu je p parametar parabole.
Tangenta na parabolu

$$y^2=2\cdot p\cdot x$$

u njezinoj točki $D(x_0, y_0)$ ima jednadžbu

$$y_0 \cdot y = p \cdot \left(x + x_0\right).$$

Kut između dva pravca

Ako su dva pravca dana eksplicitnim jednadžbama

$$\left.\begin{array}{l} y = k_1 \cdot x + l_1 \\ y = k_2 \cdot x + l_2 \end{array}\right\}$$

i ako nisu okomiti, tada za kut φ između ta dva pravca vrijedi

$$tg\ \varphi = \left|\frac{k_2 - k_1}{1 + k_1 \cdot k_2}\right|.$$

Presjek pravca i parabole dobijemo rješavanjem sustava jednadžbi:

$$\left.\begin{array}{l} 2 \cdot x + y - 12 = 0 \\ y^2 = 4 \cdot x \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = -2 \cdot x + 12 \\ y^2 = 4 \cdot x \end{array}\right\} \Rightarrow \left(-2 \cdot x + 12\right)^2 = 4 \cdot x \Rightarrow$$

$$\Rightarrow 4 \cdot x^2 - 48 \cdot x + 144 = 4 \cdot x \Rightarrow 4 \cdot x^2 - 48 \cdot x + 144 - 4 \cdot x = 0 \Rightarrow 4 \cdot x^2 - 52 \cdot x + 144 = 0 \Rightarrow$$

$$\Rightarrow 4 \cdot x^2 - 52 \cdot x + 144 = 0 \ /:4 \Rightarrow x^2 - 13 \cdot x + 36 = 0 \Rightarrow \left.\begin{array}{l} x^2 - 13 \cdot x + 36 = 0 \\ a = 1 \ , \ b = -13 \ , \ c = 36 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} a = 1 \ , \ b = -13 \ , \ c = 36 \\ x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \end{array}\right\} \Rightarrow x_{1,2} = \frac{13 \pm \sqrt{169 - 4 \cdot 1 \cdot 36}}{2 \cdot 1} \Rightarrow x_{1,2} = \frac{13 \pm \sqrt{169 - 144}}{2} \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{13 \pm \sqrt{25}}{2} \Rightarrow x_{1,2} = \frac{13 \pm 5}{2} \Rightarrow \left.\begin{array}{l} x_1 = \dfrac{13+5}{2} \\ x_2 = \dfrac{13-5}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1 = \dfrac{18}{2} \\ x_2 = \dfrac{8}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1 = 9 \\ x_2 = 4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left[\ y = -2 \cdot x + 12\ \right] \Rightarrow \left.\begin{array}{l} x_1 = 9 \ , \ y_1 = -2 \cdot x_1 + 12 \\ x_2 = 4 \ , \ y_2 = -2 \cdot x_2 + 12 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_1 = -2 \cdot 9 + 12 \\ y_2 = -2 \cdot 4 + 12 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} y_1 = -18 + 12 \\ y_2 = -8 + 12 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_1 = -6 \\ y_2 = 4 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{dirališta tangenata} \\ \text{i parabole} \end{array}\right] \Rightarrow \left.\begin{array}{l} A(4,\ 4) \\ B(9,\ -6) \end{array}\right\}.$$

Sada odredimo parametar p parabole:

$$\left.\begin{array}{l} y^2 = 4 \cdot x \\ y^2 = 2 \cdot p \cdot x \end{array}\right\} \Rightarrow 2 \cdot p = 4 \ /:2 \Rightarrow p = 2.$$

Tangente parabole u točkama A i B su:

- $$\left.\begin{array}{l} A\left(x_0,\ y_0\right) = A(4,\ 4) \ , \ p = 2 \\ y_0 \cdot y = p \cdot \left(x + x_0\right) \end{array}\right\} \Rightarrow 4 \cdot y = 2 \cdot (x + 4) \Rightarrow 4 \cdot y = 2 \cdot x + 8 \Rightarrow$$

$$\Rightarrow 4 \cdot y = 2 \cdot x + 8 \ /:4 \Rightarrow y = \frac{2}{4} \cdot x + \frac{8}{4} \Rightarrow y = \frac{1}{2} \cdot x + 2 \Rightarrow k_1 = \frac{1}{2} \ \text{ koeficijent smjera.}$$

$$\bullet \quad \left.\begin{matrix} B(x_0, y_0) = B(9, -6) \ , \ p = 2 \\ y_0 \cdot y = p \cdot (x + x_0) \end{matrix}\right\} \Rightarrow -6 \cdot y = 2 \cdot (x + 9) \Rightarrow -6 \cdot y = 2 \cdot x + 18 \Rightarrow$$

$$\Rightarrow -6 \cdot y = 2 \cdot x + 18 \ /:(-6) \Rightarrow y = -\frac{2}{6} \cdot x - \frac{18}{6} \Rightarrow y = -\frac{1}{3} \cdot x - 3 \Rightarrow k_2 = -\frac{1}{3} \text{ koeficijent smjera.}$$

Kut između tangenata parabole iznosi:

$$\left.\begin{matrix} k_1 = \frac{1}{2} \ , \ k_2 = -\frac{1}{3} \\ tg\,\varphi = \left| \frac{k_2 - k_1}{1 + k_1 \cdot k_2} \right| \end{matrix}\right\} \Rightarrow tg\,\varphi = \left| \frac{-\frac{1}{3} - \frac{1}{2}}{1 + \frac{1}{2} \cdot \left(-\frac{1}{3}\right)} \right| \Rightarrow tg\,\varphi = \left| \frac{\frac{-2-3}{6}}{1 - \frac{1}{6}} \right| \Rightarrow tg\,\varphi = \left| \frac{-\frac{5}{6}}{\frac{5}{6}} \right| \Rightarrow tg\,\varphi = |-1| \Rightarrow$$

$$\Rightarrow tg\,\varphi = 1 \Rightarrow \varphi = tg^{-1} 1 \Rightarrow \varphi = \frac{\pi}{4} \left(\text{ili } 45^0\right).$$

Vježba 090

Pravac $\frac{x}{6} + \frac{y}{12} = 1$ siječe parabolu $y^2 - 4 \cdot x = 0$. Odredi kut između tangenata parabole u tim točkama.

Rezultat: 45°.

Zadatak 091 (Irena, gimnazija)

Odredi vrh, žarište i ravnalicu parabole $2 \cdot y^2 - 5 \cdot x - 8 \cdot y - 17 = 0$.

Rješenje 091

Ponovimo!

$$(a - b)^2 = a^2 - 2 \cdot a \cdot b + b^2.$$

Parabola je skup svih točaka u ravnini čija jeudaljenost od jedne čvrste točke te ravnine jednaka udaljenosti od čvrstog pravca iste ravnine. Čvrsta točka zove se žarište ili fokus parabole, čvrsti pravac ravnalica ili direktrisa.

Parabola kojoj je os paralelna s osi x, a vrh V ima koordinate $V(x_0, y_0)$ ima jednadžbu:

$$(y - y_0)^2 = 2 \cdot p \cdot (x - x_0),$$

pri čemu je p parametar parabole.

Koordinate fokusa F su

$$F\left(x_0 + \frac{p}{2}, y_0\right),$$

a ravnalica ima jednadžbu

$$x = x_0 - \frac{p}{2}.$$

$$2 \cdot y^2 - 5 \cdot x - 8 \cdot y - 17 = 0 \Rightarrow 2 \cdot y^2 - 8 \cdot y = 5 \cdot x + 17 \Rightarrow 2 \cdot (y^2 - 4 \cdot y) = 5 \cdot x + 17 \Rightarrow$$

$$\Rightarrow 2 \cdot (y^2 - 2 \cdot 2 \cdot y) = 5 \cdot x + 17 \Rightarrow \left[\begin{matrix}\text{nadopunjavanje na} \\ \text{puni kvadrat}\end{matrix}\right] \Rightarrow 2 \cdot (y^2 - 2 \cdot 2 \cdot y + 2^2 - 2^2) = 5 \cdot x + 17 \Rightarrow$$

$$\Rightarrow 2 \cdot \left((y^2 - 2 \cdot 2 \cdot y + 2^2) - 2^2\right) = 5 \cdot x + 17 \Rightarrow 2 \cdot \left((y - 2)^2 - 4\right) = 5 \cdot x + 17 \Rightarrow$$

$$\Rightarrow 2 \cdot (y - 2)^2 - 8 = 5 \cdot x + 17 \Rightarrow 2 \cdot (y - 2)^2 = 5 \cdot x + 17 + 8 \Rightarrow 2 \cdot (y - 2)^2 = 5 \cdot x + 25 \Rightarrow$$

$$\Rightarrow 2\cdot(y-2)^2=5\cdot(x+5) \Rightarrow 2\cdot(y-2)^2=5\cdot(x+5)\ /:2 \Rightarrow (y-2)^2=\frac{5}{2}\cdot(x+5) \Rightarrow$$

$$\left.\begin{array}{l}(y-2)^2=\frac{5}{2}\cdot(x+5)\\ (y-y_0)^2=2\cdot p\cdot(x-x_0)\end{array}\right\} \Rightarrow \left.\begin{array}{l}x_0=-5\ ,\ y_0=2\\ 2\cdot p=\frac{5}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l}x_0=-5\ ,\ y_0=2\\ 2\cdot p=\frac{5}{2}\ /:4\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}x_0=-5\ ,\ y_0=2\\ \frac{p}{2}=\frac{5}{8}\end{array}\right\} \Rightarrow V(x_0,\ y_0)=V(-5,\ 2).\ \text{vrh ili tjeme parabole}$$

Fokus F iznosi:

$$\left.\begin{array}{l}x_0=-5\ ,\ y_0=2\ ,\ \frac{p}{2}=\frac{5}{8}\\ F\left(x_0+\frac{p}{2},\ y_0\right)\end{array}\right\} \Rightarrow F\left(-5+\frac{5}{8},\ 2\right)=F\left(-\frac{35}{8},\ 2\right).\ \text{žarište ili fokus parabole}$$

Ravnalica ima jednadžbu:

$$\left.\begin{array}{l}x_0=-5\ ,\ \frac{p}{2}=\frac{5}{8}\\ x=x_0-\frac{p}{2}\end{array}\right\} \Rightarrow x=-5-\frac{5}{8} \Rightarrow x=-\frac{45}{8}.\ \text{ravnalica ili direktrisa parabole}$$

Vježba 091

Odredi vrh, žarište i ravnalicu parabole $y^2 - 8 \cdot x - 4 \cdot y + 100 = 0$.

Rezultat: V(12, 2) , F(14, 2) , x = 10.

Zadatak 092 (Marinela i frendice, studentice)

Nađi jednadžbu kružnice koja dira paralelne pravce $x - 2 \cdot y + 12 = 0$ i $x - 2 \cdot y - 8 = 0$ i kružnicu $(x - 2)^2 + (y - 2)^2 = 5$.

Rješenje 092

Ponovimo!

$$(a-b)^2=a^2-2\cdot a\cdot b+b^2 \quad , \quad (a\cdot b)^n=a^n\cdot b^n \quad , \quad |a|=\begin{cases}a\ ,\ a\ge 0\\ -a\ ,\ a<0\end{cases}.$$

Udaljenost para paralelnih (usporednih) pravaca

$$p_1\ \dots\ A\cdot x+B\cdot y+C_1=0,$$
$$p_2\ \dots\ A\cdot x+B\cdot y+C_2=0$$

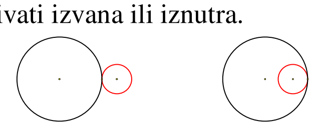
dana je formulom

$$|p_1,\ p_2|=\frac{|C_2-C_1|}{\sqrt{A^2+B^2}}.$$

Kružnica je skup svih točaka u ravnini kojima je udaljenost od zadane točke (središta) stalna. Ako je S(p, q) središte kružnice, a r njezin polumjer, tada središnja jednadžba kružnice glasi

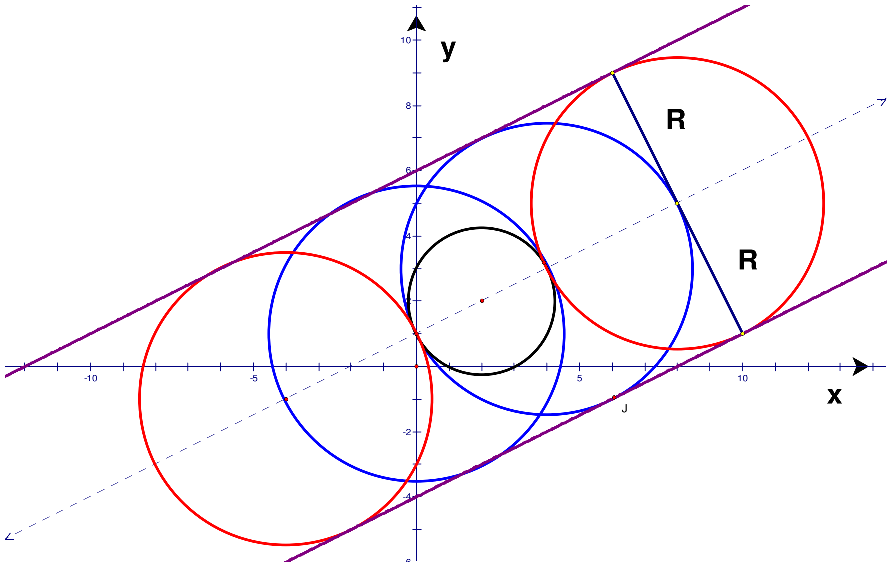
$$(x-p)^2+(y-q)^2=r^2.$$

Dvije kružnice k_1 i k_2 mogu se dodirivati izvana ili iznutra.



Neka su A(x_1, y_1) i B(x_2, y_2) dvije točke ravnine. Tada je udaljenost točaka A i B dana formulom:

$$|AB| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}.$$



Budući da su paralelni pravci x – 2 · y + 12 = 0 i x – 2 · y – 8 = 0 tangente tražene kružnice, njihova međusobna udaljenost jednaka je promjeru kružnice:

$$\left.\begin{array}{l} x - 2\cdot y + 12 = 0 \;,\; A = 1 \;,\; B = -2 \;,\; C_1 = 12 \\ x - 2\cdot y - 8 = 0 \;,\; A = 1 \;,\; B = -2 \;,\; C_2 = -8 \end{array}\right\} \Rightarrow \left[2\cdot R = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}} \right] \Rightarrow 2\cdot R = \frac{|-8-12|}{\sqrt{1^2 + (-2)^2}} \Rightarrow$$

$$\Rightarrow 2\cdot R = \frac{|-20|}{\sqrt{1+4}} \Rightarrow 2\cdot R = \frac{20}{\sqrt{5}} \;/:2 \Rightarrow R = \frac{10}{\sqrt{5}} \Rightarrow \left[\begin{array}{l}\text{racionalizacija} \\ \text{nazivnika}\end{array}\right] \Rightarrow R = \frac{10}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} \Rightarrow$$

$$\Rightarrow R = \frac{10\cdot\sqrt{5}}{5} \Rightarrow R = 2\cdot\sqrt{5}. \;\text{polumjer tražene kružnice}$$

Središte S(p, q) tražene kružnice nalazi se na pravcu koji je paralelan s pravcima x – 2 · y + 12 = 0 i x – 2 · y – 8 = 0, a jednako je od njih udaljen za polumjer R kružnice. Tražimo jednadžbu tog pravca:

$$\left.\begin{array}{l} x - 2\cdot y + 12 = 0 \;,\; A = 1 \;,\; B = -2 \;,\; C_1 = 12 \\ x - 2\cdot y + C = 0 \;,\; A = 1 \;,\; B = -2 \;,\; C_2 = C \;\; \text{traženi pravac} \end{array}\right\} \Rightarrow \left[R = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}} \right] \Rightarrow$$

$$\Rightarrow 2\cdot\sqrt{5} = \frac{|C-12|}{\sqrt{1^2 + (-2)^2}} \Rightarrow 2\cdot\sqrt{5} = \frac{|C-12|}{\sqrt{1+4}} \Rightarrow 2\cdot\sqrt{5} = \frac{|C-12|}{\sqrt{5}} \;/\cdot\sqrt{5} \Rightarrow$$

$$\Rightarrow |C-12| = 2\cdot\left(\sqrt{5}\right)^2 \Rightarrow |C-12| = 2\cdot 5 \Rightarrow |C-12| = 10 \Rightarrow \left.\begin{array}{l} C - 12 = 10 \\ C - 12 = -10 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} C_1 = 10 + 12 \\ C_2 = -10 + 12 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1 = 22 \;\; \text{nema smisla jer taj pravac ne leži izmeđ dva zadana} \\ C_2 = 2 \end{array}\right\} \Rightarrow$$

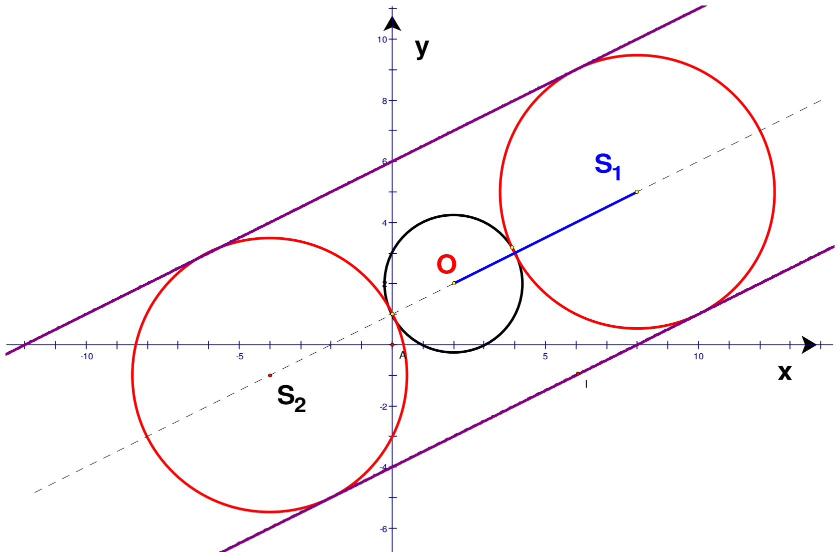
$$\Rightarrow x - 2\cdot y + 2 = 0. \;\text{traženi pravac}$$

Točka S(p, q), središte tražene kružnice, leži na pravcu x – 2 · y + 2 = 0 pa vrijedi:

$$\left.\begin{array}{l} S(x,\ y)=S(p,\ q) \\ x-2\cdot y+2=0 \end{array}\right\} \Rightarrow\ p-2\cdot q+2=0\ \Rightarrow\ p=2\cdot q-2.$$

Koordinate točke S su:

$$S(2\cdot q-2,\ q).$$



Budući da se tražena kružnica

$$(x-p)^2+(y-q)^2=\left(2\cdot\sqrt{5}\right)^2\ ,\ R=2\cdot\sqrt{5}$$

i kružnica

$$(x-2)^2+(y-2)^2=5\ ,\ r=\sqrt{5}$$

dodiruju izvana, za udaljenost između njihovih središta O i S vrijedi:

$$\left.\begin{array}{l} O(x_1,\ y_1)=O(2,\ 2)\ ,\ S(x_2,\ y_2)=S(2\cdot q-2,\ 2) \\ |OS|=r+R=\sqrt{5}+2\cdot\sqrt{5}=3\cdot\sqrt{5} \end{array}\right\} \Rightarrow \left[\ |OS|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\ \right] \Rightarrow$$

$$\Rightarrow\ |OS|=\sqrt{(2\cdot q-2-2)^2+(q-2)^2}\ \Rightarrow\ 3\cdot\sqrt{5}=\sqrt{(2\cdot q-4)^2+(q-2)^2}\ /^2\ \Rightarrow$$

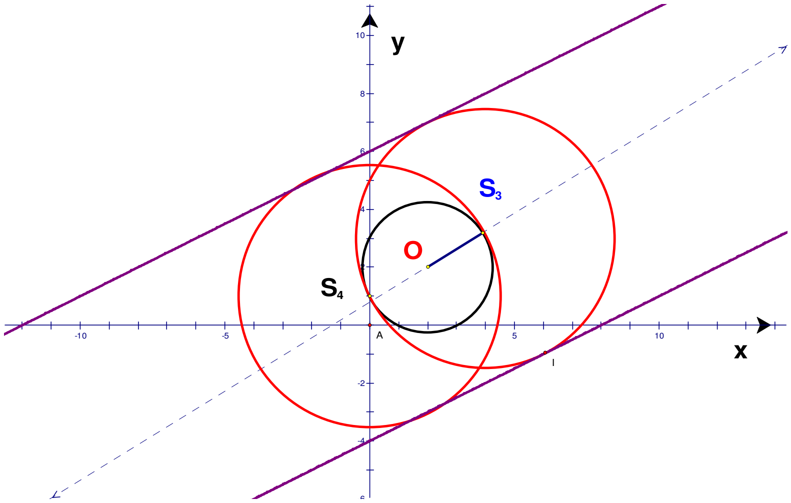
$$\Rightarrow\ 45=(2\cdot q-4)^2+(q-2)^2\ \Rightarrow\ (2\cdot q-4)^2+(q-2)^2=45\ \Rightarrow\ (2\cdot(q-2))^2+(q-2)^2=45\ \Rightarrow$$

$$\Rightarrow\ 4\cdot(q-2)^2+(q-2)^2=45\ \Rightarrow\ 5\cdot(q-2)^2=45\ /:5\ \Rightarrow\ (q-2)^2=9\ /\sqrt{\ }\ \Rightarrow\ q-2=\pm\sqrt{9}\ \Rightarrow$$

$$\Rightarrow\ q-2=\pm 3\ \Rightarrow \left.\begin{array}{l} q-2=3 \\ q-2=-3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} q=3+2 \\ q=-3+2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} q_1=5 \\ q_2=-1 \end{array}\right\} \Rightarrow [\ p=2\cdot q-2\ ] \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} p_1=2\cdot q_1-2 \\ p_2=2\cdot q_2-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_1=2\cdot 5-2 \\ p_2=2\cdot(-1)-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_1=10-2 \\ p_2=-2-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_1=8 \\ p_2=-4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} S_1(p_1,\ q_1)=S_1(8,\ 5) \\ S_2(p_2,\ q_2)=S_2(-4,\ -4) \end{array}\right\}.\ \text{središta traženih kružnica (dva rješenja)}$$



Budući da se tražena kružnica

$$\left(x-p\right)^2+\left(y-q\right)^2=\left(2\cdot\sqrt{5}\right)^2 \quad , \quad R=2\cdot\sqrt{5}$$

i kružnica

$$\left(x-2\right)^2+\left(y-2\right)^2=5 \quad , \quad r=\sqrt{5}$$

dodiruju iznutra, za udaljenost između njihovih središta O i S vrijedi:

$$\left.\begin{array}{l} O\left(x_1,\ y_1\right)=O\left(2,\ 2\right) \ , \ S\left(x_2,\ y_2\right)=S\left(2\cdot q-2,\ 2\right) \\ \left|OS\right|=R-r=2\cdot\sqrt{5}-\sqrt{5}=\sqrt{5} \end{array}\right\} \Rightarrow \left[\left|OS\right|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}\right] \Rightarrow$$

$$\Rightarrow \left|OS\right|=\sqrt{\left(2\cdot q-2-2\right)^2+\left(q-2\right)^2} \Rightarrow \sqrt{5}=\sqrt{\left(2\cdot q-4\right)^2+\left(q-2\right)^2} \ /^2 \Rightarrow$$

$$\Rightarrow 5=\left(2\cdot q-4\right)^2+\left(q-2\right)^2 \Rightarrow \left(2\cdot q-4\right)^2+\left(q-2\right)^2=5 \Rightarrow \left(2\cdot\left(q-2\right)\right)^2+\left(q-2\right)^2=5 \Rightarrow$$

$$\Rightarrow 4\cdot\left(q-2\right)^2+\left(q-2\right)^2=5 \Rightarrow 5\cdot\left(q-2\right)^2=5 \ /:5 \Rightarrow \left(q-2\right)^2=1 \ /\sqrt{\ } \Rightarrow q-2=\pm\sqrt{1} \Rightarrow$$

$$\Rightarrow q-2=\pm 1 \Rightarrow \left.\begin{array}{l} q-2=1 \\ q-2=-1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} q=1+2 \\ q=-1+2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} q_3=3 \\ q_4=1 \end{array}\right\} \Rightarrow \left[\ p=2\cdot q-2\ \right] \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} p_3=2\cdot q_3-2 \\ p_4=2\cdot q_4-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_3=2\cdot 3-2 \\ p_4=2\cdot 1-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_3=6-2 \\ p_2=2-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} p_3=4 \\ p_4=0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} S_3\left(p_3,\ q_3\right)=S_3\left(4,\ 3\right) \\ S_4\left(p_4,\ q_4\right)=S_4\left(0,\ 1\right) \end{array}\right\}. \ \text{središta traženih kružnica (dva rješenja)}$$

Ukupno postoje 4 kružnice koje ispunjavaju uvjete:

$$\left(x-8\right)^2+\left(y-5\right)^2=20 \quad , \quad \left(x+4\right)^2+\left(y+4\right)^2=20$$

$$\left(x-4\right)^2+\left(y-3\right)^2=20 \quad , \quad x^2+\left(y-1\right)^2=20.$$

Vježba 092

Nađi jednadžbu kružnice koja dira paralelne pravce $x-2\cdot y+12=0$ i $x-2\cdot y-8=0$ i kružnicu $x^2+y^2-4\cdot x-4\cdot y+3=0$.

Rezultat: $(x-8)^2+(y-5)^2=20$, $(x+4)^2+(y+4)^2=20$
$(x-4)^2+(y-3)^2=20$, $x^2+(y-1)^2=20$.

Zadatak 093 (Mirela, srednja škola)

Parabola $y^2 = 2 \cdot p \cdot x$ ima žarište F(1, 0). Odredi joj diralište s tangentom koja s osi apscisa zatvara kut 135°.

Rješenje 093

Ponovimo!

Parabola kojoj se os podudara s x – osi koordinatnog sustava, vrh sa ishodištem, a fokus se nalazi na pozitivnom dijelu x – osi ima jednadžbu

$$y^2 = 2 \cdot p \cdot x,$$

pri čemu je p parametar parabole.

Pravac

$$y = k \cdot x + l$$

dodiruje parabolu

$$y^2 = 2 \cdot p \cdot x$$

ako i samo ako vrijedi

$$p = 2 \cdot k \cdot l.$$

Koordinate dirališta D parabole

$$y^2 = 2 \cdot p \cdot x$$

i tangente

$$y = k \cdot x + l$$

glase:

$$D\left(\frac{l}{k},\ 2 \cdot l\right).$$

Eksplicitni oblik jednadžbe pravca glasi

$$y = k \cdot x + l,$$

gdje je k koeficijent smjera pravca, a l odsječak pravca na osi y. Za koeficijent smjera k vrijedi

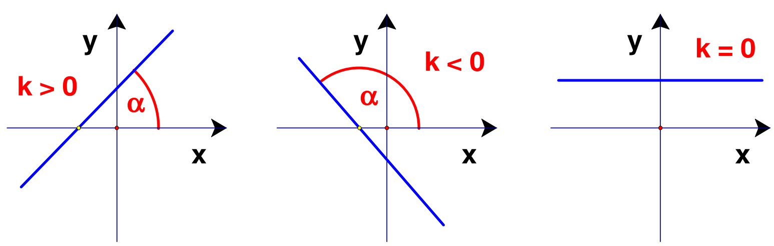
$$k = tg\ \alpha,$$

gdje je α kut koji pravac zatvara s pozitivnim smjerom osi x. Uočimo:

1) ako je α šiljasti kut, onda je $k > 0$,

2) ako je α tupi kut, onda je $k < 0$,

3) ako je $\alpha = 0$, onda je $k = 0$, pravac je paralelan s osi x.



Parametar p parabole iznosi:

$$F\left(\frac{p}{2},\ 0\right) = F(1,\ 0) \Rightarrow \frac{p}{2} = 1\ /\cdot 2 \Rightarrow p = 2.$$

Budući da tangenta parabole s osi apscisa zatvara kut 135° , njezin koeficijent smjera k ima vrijednost:

$$\left.\begin{array}{l} \alpha = 135^0 \\ k = tg\ \alpha \end{array}\right\} \Rightarrow k = tg\ 135^0 \Rightarrow k = -1.$$

Iz uvjeta dodira pravca (tangente) i parabole dobije se l odsječak pravca na osi y:

$$\left.\begin{array}{l} k = -1\ ,\ p = 2 \\ p = 2 \cdot k \cdot l \end{array}\right\} \Rightarrow 2 = 2 \cdot (-1) \cdot l \Rightarrow 2 = -2 \cdot l \Rightarrow 2 \cdot l = -2\ /:2 \Rightarrow k = -1.$$

Dirалište D tangente i parabole ima koordinate:

$$\left.\begin{array}{l} k=-1 \;,\; l=-1 \\ D\left(\dfrac{l}{k},\, 2\cdot l\right) \end{array}\right\} \Rightarrow D\left(\frac{-1}{-1},\, 2\cdot(-1)\right)=D(1,\,-2).$$

Vježba 093

Parabola $y^2 = 2 \cdot p \cdot x$ ima žarište F(1, 0). Odredi joj diralište s tangentom koja s osi apscisa zatvara kut 45°.

Rezultat: D(1, 2).

Zadatak 094 (Mirela, srednja škola)

Kolika je međusobna udaljenost tangenata elipse $3 \cdot x^2 + 4 \cdot y^2 = 48$ koje su paralelne s pravcem $x - 2 \cdot y + 12 = 0$?

Rješenje 094

Ponovimo!

$$|a|=\begin{cases} a \;,\; a\ge 0 \\ -a \;,\; a<0 \end{cases}.$$

Elipsa kojoj se središte podudara s ishodištem koordinatnog sustava, smjer glavne osi s x-osi, a smjer sporedne osi s y-osi ima jednadžbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow b^2\cdot x^2+a^2\cdot y^2=a^2\cdot b^2,$$

gdje su a i b velika i mala poluos.
Uvjet dodira pravca i elipse
Pravac

$$y=k\cdot x+l$$

dodiruje elipsu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ako i samo ako vrijedi

$$a^2\cdot k^2+b^2=l^2.$$

Uvjet usporednosti (paralelnosti) pravaca:
Ako su pravci dani eksplicitnim jednadžbama $y = k_1 \cdot x + l_1$, $y = k_2 \cdot x + l_2$, $k_1, k_2 \neq 0$, tada su usporedni (paralelni) ako i samo ako je

$$k_1 = k_2.$$

Eksplicitni oblik jednadžbe pravca

$$y=k\cdot x+l \Rightarrow \left.\begin{array}{l} k - \text{koeficijent smjera} \\ l - \text{odsječak na osi } y \end{array}\right\}$$

Udaljenost para paralelnih (usporednih) pravaca

$$p_1 \;\ldots\; A\cdot x+B\cdot y+C_1=0,$$
$$p_2 \;\ldots\; A\cdot x+B\cdot y+C_2=0$$

dana je formulom

$$|p_1,\, p_2|=\frac{|C_2-C_1|}{\sqrt{A^2+B^2}}.$$

Da bismo odredili a^2 i b^2 treba jednadžbu elipse svesti na oblik

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

1.inačica

$$3\cdot x^2+4\cdot y^2=48 \;/:48 \Rightarrow \frac{3\cdot x^2}{48}+\frac{4\cdot y^2}{48}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{12}=1 \Rightarrow \left.\begin{array}{l} a^2=16 \\ b^2=12 \end{array}\right\}.$$

2.inačica

Rabimo koordinate tjemena elipse

$$\left.\begin{array}{l} 3\cdot x^2+4\cdot y^2=48 \\ A(x,\ y)=A(a,\ 0) \end{array}\right\} \Rightarrow 3\cdot a^2+4\cdot 0^2=48 \Rightarrow 3\cdot a^2+0=48 \Rightarrow 3\cdot a^2=48 \;/:3 \Rightarrow a^2=16.$$

$$\left.\begin{array}{l} 3\cdot x^2+4\cdot y^2=48 \\ D(x,\ y)=D(0,\ b) \end{array}\right\} \Rightarrow 3\cdot 0^2+4\cdot b^2=48 \Rightarrow 0+4\cdot b^2=48 \Rightarrow 4\cdot b^2=48 \;/:4 \Rightarrow b^2=12.$$

Koeficijent smjera k zadanog pravca iznosi:

$$x-2\cdot y+12=0 \Rightarrow -2\cdot y=-x-12 \;/:(-2) \Rightarrow y=\frac{1}{2}\cdot x+6 \Rightarrow k=\frac{1}{2}.$$

Budući da su tangente paralelne sa zadanim pravcem, imaju isti koeficijent smjera:

$$k=\frac{1}{2}.$$

Odsječak l traženih tangenata na osi y dobije se iz uvjeta dodira pravca i elipse:

$$\left.\begin{array}{l} a^2=16\ ,\ b^2=12\ ,\ k=\frac{1}{2} \\ a^2\cdot k^2+b^2=l^2 \end{array}\right\} \Rightarrow 16\cdot\left(\frac{1}{2}\right)^2+12=l^2 \Rightarrow 16\cdot\frac{1}{4}+12=l^2 \Rightarrow 4+12=l^2 \Rightarrow$$

$$\Rightarrow l^2=16 \;/\sqrt{\ } \Rightarrow l_{1,2}=\pm\sqrt{16} \Rightarrow \left.\begin{array}{l} l_1=4 \\ l_2=-4 \end{array}\right\}.$$

Jednadžbe traženih tangenata glase:

$$\left.\begin{array}{l} k=\frac{1}{2}\ ,\ l_1=4\ ,\ y=k\cdot x+l_1 \\ k=\frac{1}{2}\ ,\ l_2=-4\ ,\ y=k\cdot x+l_2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=\frac{1}{2}\cdot x+4 \;/\cdot 2 \\ y=\frac{1}{2}\cdot x-4 \;/\cdot 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\cdot y=x+8 \\ 2\cdot y=x-8 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-2\cdot y+8=0 \\ x-2\cdot y-8=0 \end{array}\right\}.$$

Udaljenost d između tangenata elipse iznosi:

$$\left.\begin{array}{l} x-2\cdot y+8=0\ ,\ A=1\ ,\ B=-2\ ,\ C_1=8 \\ x-2\cdot y-8=0\ ,\ A=1\ ,\ B=-2\ ,\ C_2=-8 \end{array}\right\} \Rightarrow \left[d=\frac{|C_2-C_1|}{\sqrt{A^2+B^2}}\right] \Rightarrow d=\frac{|-8-8|}{\sqrt{1^2+(-2)^2}} \Rightarrow$$

$$\Rightarrow d=\frac{|-16|}{\sqrt{1+4}} \Rightarrow d=\frac{16}{\sqrt{5}} \Rightarrow \left[\begin{array}{l} \text{racionalizacija} \\ \text{nazivnika} \end{array}\right] \Rightarrow d=\frac{16}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} \Rightarrow d=\frac{16\cdot\sqrt{5}}{5}.$$

Vježba 094

Kolika je međusobna udaljenost tangenata elipse $3\cdot x^2+4\cdot y^2=48$ koje su paralelne s pravcem $\frac{x}{-12}+\frac{y}{6}=1$?

Rezultat: $\frac{16\cdot\sqrt{5}}{5}.$

Zadatak 095 (Marko, srednja škola)

Odredi jednadžbu hiperbole kojoj su žarišta $F_1(-10, 0)$, $F_2(10, 0)$ i asimptote $y = \pm 2\cdot x$.

Rješenje 095

Ponovimo!

Hiperbola kojoj se središte podudara s ishodištem koordinatnog sustava, a fokusi leže na osi x ima jednadžbu

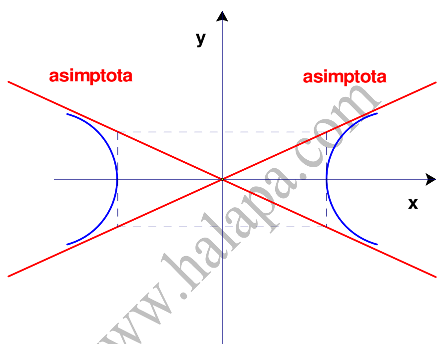
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow b^2\cdot x^2-a^2\cdot y^2=a^2\cdot b^2,$$

gdje je a realna poluos, b imaginarna poluos.
Koordinate žarišta ili fokusa hiperbole glase

$$F_1(-e, 0) \quad , \quad F_2(e, 0)$$

gdje je e linearni ekscentricitet

$$e^2=a^2+b^2.$$



Pravci koji sadrže dijagonale središnjeg pravokutnika s dimenzijama $2\cdot a$ i $2\cdot b$ zovu se asimptote i njihove su jednadžbe:

$$y=-\frac{b}{a}\cdot x \quad , \quad y=\frac{b}{a}\cdot x.$$

Budući da je pravac $y = 2\cdot x$ asimptota hiperbole, dobije se:

$$\left.\begin{array}{l} y=2\cdot x \\ y=\frac{b}{a}\cdot x \end{array}\right\} \Rightarrow \frac{b}{a}=2 \;/\cdot a \Rightarrow b=2\cdot a.$$

Za fokus hiperbole vrijedi:

$$\left.\begin{array}{l} F_2(10, 0) \\ F_2(e, 0) \end{array}\right\} \Rightarrow e=10 \Rightarrow \left.\begin{array}{l} a^2+b^2=e^2 \\ e=10 \end{array}\right\} \Rightarrow a^2+b^2=10^2 \Rightarrow a^2+b^2=100.$$

Da bismo odredili a^2 i b^2 riješimo sustav jednadžbi:

$$\left.\begin{array}{l} a^2+b^2=100 \\ b=2\cdot a \end{array}\right\} \Rightarrow \left.\begin{array}{l} a^2+b^2=100 \\ b=2\cdot a \;/^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a^2+b^2=100 \\ b^2=4\cdot a^2 \end{array}\right\} \Rightarrow a^2+4\cdot a^2=100 \Rightarrow$$

$$\Rightarrow 5\cdot a^2=100 \;/:5 \Rightarrow a^2=20 \Rightarrow \left.\begin{array}{l} a^2=20 \\ b^2=4\cdot a^2 \end{array}\right\} \Rightarrow b^2=4\cdot 20 \Rightarrow b^2=80.$$

Jednadžba hiperbole glasi:

$$\left.\begin{array}{l} a^2 = 20 \ , \ b^2 = 80 \\ b^2 \cdot x^2 - a^2 \cdot y^2 = a^2 \cdot b^2 \end{array}\right\} \Rightarrow 80 \cdot x^2 - 20 \cdot y^2 = 80 \cdot 20 \ /: 20 \ \Rightarrow \ 4 \cdot x^2 - y^2 = 80.$$

Vježba 095

Odredi jednadžbu hiperbole kojoj su žarišta $F_1(-5, 0)$, $F_2(5, 0)$ i asimptote $y = \pm \frac{3}{4} \cdot x$.

Rezultat: $9 \cdot x^2 - 16 \cdot y^2 = 144.$

Zadatak 096 (Marko, srednja škola)

Odredi jednadžbu hiperbole kojoj je pravac $x + 2 \cdot y = 0$ asimptota, a pravac $5 \cdot x - 6 \cdot y - 8 = 0$ tangenta.

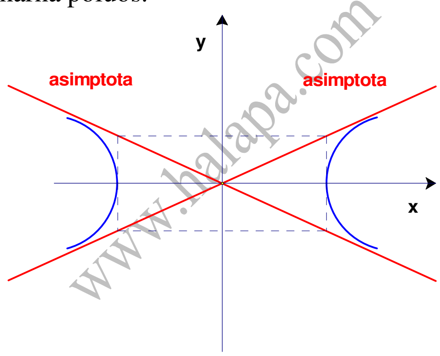
Rješenje 096

Ponovimo!

Hiperbola kojoj se središte podudara s ishodištem koordinatnog sustava, a fokusi leže na osi x ima jednadžbu

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ \Rightarrow \ b^2 \cdot x^2 - a^2 \cdot y^2 = a^2 \cdot b^2,$$

gdje je a realna poluos, b imaginarna poluos.



Pravci koji sadrže dijagonale središnjeg pravokutnika s dimenzijama $2 \cdot a$ i $2 \cdot b$ zovu se asimptote i njihove jednadžbe glase:

$$y = -\frac{b}{a} \cdot x \quad , \quad y = \frac{b}{a} \cdot x.$$

Uvjet dodira pravca i hiperbole

Pravac

$$y = k \cdot x + l$$

dodiruje hiperbolu

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

ako i samo ako vrijedi

$$a^2 \cdot k^2 - b^2 = l^2.$$

Eksplicitni oblik jednadžbe pravca

$$y = k \cdot x + l \ \Rightarrow \ \left.\begin{array}{l} k - \text{koeficijent smjera} \\ l - \text{odsječak na osi } y \end{array}\right\}.$$

Budući da je pravac $x + 2 \cdot y = 0$ asimptota hiperbole, vrijedi:

$$x+2\cdot y=0 \Rightarrow 2\cdot y=-x \ /:2 \Rightarrow y=-\frac{1}{2}\cdot x \Rightarrow \left[y=-\frac{b}{a}\cdot x\right] \Rightarrow -\frac{b}{a}=-\frac{1}{2} \ /\cdot(-2\cdot a) \Rightarrow$$

$$\Rightarrow 2\cdot b=a \Rightarrow a=2\cdot b.$$

Jednadžbu zadane tangente napišemo u eksplicitnom obliku kako bismo odredili k koeficijent smjera i l odsječak na osi y:

$$5\cdot x-6\cdot y-8=0 \Rightarrow -6\cdot y=-5\cdot x+8 \ /:(-6) \Rightarrow y=\frac{5}{6}\cdot x-\frac{8}{6} \Rightarrow y=\frac{5}{6}\cdot x-\frac{4}{3} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} k=\frac{5}{6} \quad \text{koeficijent smjera} \\ l=-\frac{4}{3} \ \text{odsječak na osi } y \end{array}\right\}.$$

Iz uvjeta dodira pravca i hiperbole dobije se:

$$\left.\begin{array}{l} k=\frac{5}{6} \ , \ l=-\frac{4}{3} \\ a^2\cdot k^2-b^2=l^2 \end{array}\right\} \Rightarrow a^2\cdot\left(\frac{5}{6}\right)^2-b^2=\left(-\frac{4}{3}\right)^2 \Rightarrow \frac{25}{36}\cdot a^2-b^2=\frac{16}{9} \ /\cdot 36 \Rightarrow$$

$$\Rightarrow 25\cdot a^2-36\cdot b^2=64.$$

Da bismo odredili a^2 i b^2 riješimo sustav jednadžbi:

$$\left.\begin{array}{l} 25\cdot a^2-36\cdot b^2=64 \\ a=2\cdot b \end{array}\right\} \Rightarrow \left.\begin{array}{l} 25\cdot a^2-36\cdot b^2=64 \\ a=2\cdot b \ /^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 25\cdot a^2-36\cdot b^2=64 \\ a^2=4\cdot b^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 25\cdot 4\cdot b^2-36\cdot b^2=64 \Rightarrow 100\cdot b^2-36\cdot b^2=64 \Rightarrow 64\cdot b^2=64 \ /:64 \Rightarrow b^2=1 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} b^2=1 \\ a^2=4\cdot b^2 \end{array}\right\} \Rightarrow a^2=4\cdot 1 \Rightarrow a^2=4.$$

Jednadžba hiperbole glasi:

$$\left.\begin{array}{l} a^2=4 \ , \ b^2=1 \\ b^2\cdot x^2-a^2\cdot y^2=a^2\cdot b^2 \end{array}\right\} \Rightarrow 1\cdot x^2-4\cdot y^2=4\cdot 1 \Rightarrow x^2-4\cdot y^2=4.$$

Vježba 096

Odredi jednadžbu hiperbole kojoj je pravac x – 2 · y = 0 asimptota, a pravac 5 · x – 6 · y – 8 = 0 tangenta.

Rezultat: $x^2-4\cdot y^2=4.$

Zadatak 097 (Martina, ekonomska škola)

Odredi jednadžbe zajedničkih tangenata hiperbole 20 · x^2 – 36 · y^2 = 720 i kružnice x^2 + y^2 = 8.

Rješenje 097

Ponovimo!

Hiperbola čiji fokusi leže na x – osi, a središte joj je ishodište koordinatnog sustava dana je jednadžbom

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

gdje je a realna poluos, b imaginarna poluos.
Pravac

$$y = k \cdot x + l$$

dodiruje hiperbolu

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

ako i samo ako vrijedi:

$$a^2 \cdot k^2 - b^2 = l^2.$$

Ako je S(0, 0) središte kružnice, a r njezin polumjer, tada središnja jednadžba kružnice glasi:

$$x^2 + y^2 = r^2.$$

Kružnica

$$x^2 + y^2 = r^2$$

i pravac

$$y = k \cdot x + l$$

dodiruju se ako i samo ako vrijedi:

$$r^2 \cdot \left(1 + k^2\right) = l^2.$$

$$\left.\begin{array}{l} 20 \cdot x^2 - 36 \cdot y^2 = 720 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 20 \cdot x^2 - 36 \cdot y^2 = 720 \;/ : 720 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{20 \cdot x^2}{720} - \frac{36 \cdot y^2}{720} = 1 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \frac{x^2}{36} - \frac{y^2}{20} = 1 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a^2 = 36 \\ b^2 = 20 \end{array}\right\}.$$

Postavimo uvjet da pravac

$$y = k \cdot x + l$$

bude tangenta zadane hiperbole:

$$\left.\begin{array}{l} a^2 = 36 \;,\; b^2 = 20 \\ a^2 \cdot k^2 - b^2 = l^2 \end{array}\right\} \Rightarrow 36 \cdot k^2 - 20 = l^2.$$

Postavimo uvjet da pravac

$$y = k \cdot x + l$$

bude tangenta zadane kružnice:

$$\left.\begin{array}{l} x^2 + y^2 = 8 \\ r^2 \cdot \left(1 + k^2\right) = l^2 \end{array}\right\} \Rightarrow 8 \cdot \left(1 + k^2\right) = l^2.$$

Iz sustava od dvije kvadratne jednadžbe sa dvije nepoznanice dobije se k i l.

$$\left.\begin{array}{l} 36\cdot k^2-20=l^2 \\ 8\cdot\left(1+k^2\right)=l^2 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{komparacije} \end{array}\right] \Rightarrow 36\cdot k^2-20=8\cdot\left(1+k^2\right) \Rightarrow 36\cdot k^2-20=8+8\cdot k^2 \Rightarrow$$

$$\Rightarrow 36\cdot k^2-8\cdot k^2=8+20 \Rightarrow 28\cdot k^2=28 \Rightarrow 28\cdot k^2=28 \; /:28 \Rightarrow k^2=1 \Rightarrow k^2=1 \; /\sqrt{\;} \Rightarrow$$

$$\Rightarrow k_{1,2}=\pm\sqrt{1} \Rightarrow \left.\begin{array}{l} k_1=1 \\ k_2=-1 \end{array}\right\}.$$

Sada računamo l odsječak na y osi.

$$\left.\begin{array}{l} 8\cdot\left(1+k^2\right)=l^2 \\ k^2=1 \end{array}\right\} \Rightarrow 8\cdot(1+1)=l^2 \Rightarrow 8\cdot 2=l^2 \Rightarrow 16=l^2 \Rightarrow l^2=16 \Rightarrow l^2=16 \; /\sqrt{\;} \Rightarrow$$

$$\Rightarrow l_{1,2}=\pm\sqrt{16} \Rightarrow \left.\begin{array}{l} l_1=4 \\ l_2=-4 \end{array}\right\}.$$

Jednadžbe zajedničkih tangenata hiperbole $20\cdot x^2-36\cdot y^2=720$ i kružnice $x^2+y^2=8$ glase:

- $\left.\begin{array}{l} k_1=1 \;,\; l_1=4 \\ y=k_1\cdot x+l_1 \end{array}\right\} \Rightarrow y=x+4,$
- $\left.\begin{array}{l} k_1=1 \;,\; l_2=-4 \\ y=k_1\cdot x+l_2 \end{array}\right\} \Rightarrow y=x-4,$
- $\left.\begin{array}{l} k_2=-1 \;,\; l_1=4 \\ y=k_2\cdot x+l_1 \end{array}\right\} \Rightarrow y=-x+4,$
- $\left.\begin{array}{l} k_2=-1 \;,\; l_2=-4 \\ y=k_2\cdot x+l_2 \end{array}\right\} \Rightarrow y=-x-4.$

Vježba 097

Odredi jednadžbe zajedničkih tangenata hiperbole $10\cdot x^2-18\cdot y^2=360$ i kružnice $x^2+y^2=8$.

Rezultat: $y=x+4 \;,\; y=x-4 \;,\; y=-x+4 \;,\; y=-x-4.$

Zadatak 098 (Martina, ekonomska škola)

Sjecišta elipsa $x^2+16\cdot y^2=64$ i $3\cdot x^2+8\cdot y^2=72$ su vrhovi pravokutnika. Koliko iznosi površina tog pravokutnika?

Rješenje 098

Ponovimo!

Elipsa kojoj se središte podudara s ishodištem koordinatnog sustava, a fokusi leže na osi x ima jednadžbu

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow b^2\cdot x^2+a^2\cdot y^2=a^2\cdot b^2,$$

gdje je a velika poluos, b mala poluos.

Površina pravokutnika duljina stranica a i b iznosi:

$$P=a\cdot b.$$

Da bismo našli sjecišta elipsa moramo riješiti sustav od dvije kvadratne jednadžbe sa dvije nepoznanice. Općenito mogu biti 4 rješenja.

$$\left.\begin{matrix} x^2+16\cdot y^2=64 \\ 3\cdot x^2+8\cdot y^2=72 \end{matrix}\right\} \Rightarrow \left[\begin{matrix}\text{metoda suprotnih} \\ \text{koeficijenata}\end{matrix}\right] \Rightarrow \left.\begin{matrix} x^2+16\cdot y^2=64 \\ 3\cdot x^2+8\cdot y^2=72 \; / \cdot(-2) \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} x^2+16\cdot y^2=64 \\ -6\cdot x^2-16\cdot y^2=-144 \end{matrix}\right\} \Rightarrow -5\cdot x^2=-80 \Rightarrow -5\cdot x^2=-80 \; /:(-5) \Rightarrow x^2=16 \Rightarrow$$

$$\Rightarrow x^2=16 \; /\sqrt{} \Rightarrow x_{1,2}=\pm\sqrt{16} \Rightarrow \left.\begin{matrix} x_1=4 \\ x_2=-4 \end{matrix}\right\}.$$

Sada računamo pripadne y.

$x_1 = 4$

$$\left.\begin{matrix} x=4 \Rightarrow x^2=16 \\ x^2+16\cdot y^2=64 \end{matrix}\right\} \Rightarrow 16+16\cdot y^2=64 \Rightarrow 16\cdot y^2=64-16 \Rightarrow 16\cdot y^2=48 \Rightarrow$$

$$\Rightarrow 16\cdot y^2=48 \; /:16 \Rightarrow y^2=3 \Rightarrow y^2=3 \; /\sqrt{} \Rightarrow y_{1,2}=\pm\sqrt{3} \Rightarrow \left.\begin{matrix} y_1=\sqrt{3} \\ y_2=-\sqrt{3} \end{matrix}\right\}.$$

Točke presjeka su:

$$A\left(4, \sqrt{3}\right) \; , \; B\left(4, -\sqrt{3}\right).$$

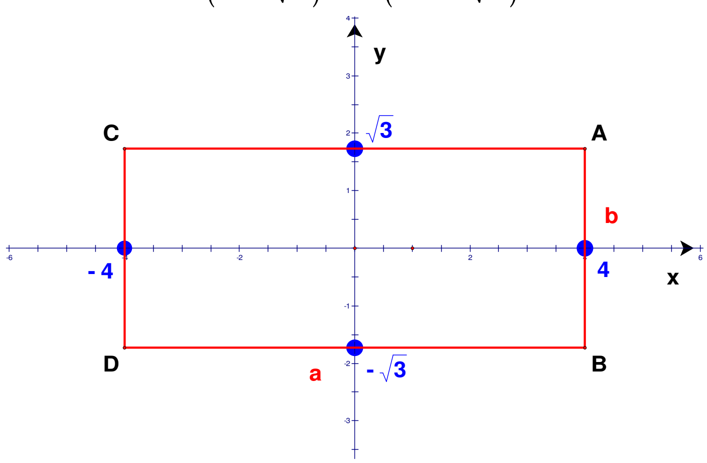
$x_2 = -4$

$$\left.\begin{matrix} x=-4 \Rightarrow x^2=16 \\ x^2+16\cdot y^2=64 \end{matrix}\right\} \Rightarrow 16+16\cdot y^2=64 \Rightarrow 16\cdot y^2=64-16 \Rightarrow 16\cdot y^2=48 \Rightarrow$$

$$\Rightarrow 16\cdot y^2=48 \; /:16 \Rightarrow y^2=3 \Rightarrow y^2=3 \; /\sqrt{} \Rightarrow y_{3,4}=\pm\sqrt{3} \Rightarrow \left.\begin{matrix} y_3=\sqrt{3} \\ y_4=-\sqrt{3} \end{matrix}\right\}.$$

Točke presjeka su:

$$C\left(-4, \sqrt{3}\right) \; , \; D\left(-4, -\sqrt{3}\right).$$



Sa slike vidi se da su duljine stranica pravokutnika

$$a=4+4 \Rightarrow a=8 \quad , \quad b=\sqrt{3}+\sqrt{3} \Rightarrow b=2\cdot\sqrt{3}$$

pa njegova površina iznosi:

$$P = a \cdot b \Rightarrow P = 8 \cdot 2 \cdot \sqrt{3} \Rightarrow P = 16 \cdot \sqrt{3}.$$

Vježba 098

Sjecišta elipsa $x^2 + 16 \cdot y^2 = 64$ i $3 \cdot x^2 + 8 \cdot y^2 = 72$ su vrhovi pravokutnika. Koliko iznosi opseg tog pravokutnika?

Rezultat: $16 + 4 \cdot \sqrt{3}.$

Zadatak 099 (Marko, gimnazija)

Asimptota hiperbole je pravac $y = 2 \cdot x$. Na hiperboli je točka (5, 8). Jednadžba hiperbole je:

$$A. \ \frac{x^2}{36} - \frac{y^2}{9} = 1 \quad , \quad B. \ \frac{x^2}{9} - \frac{y^2}{36} = 1 \quad , \quad C. \ \frac{x^2}{6} - \frac{y^2}{3} = 1 \quad , \quad D. \ \frac{x^2}{3} - \frac{y^2}{6} = 1$$

Rješenje 099

Ponovimo!

$$\frac{a-b}{n} = \frac{a}{n} - \frac{b}{n}.$$

Hiperbola čiji fokusi leže na x – osi, a središte joj je ishodište koordinatnog sustava dana je jednadžbom

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

gdje je a realna poluos, b imaginarna poluos.
Pravac koji sadrži dijagonale središnjeg pravokutnika dimenzija 2 · a i 2 · b nazivamo asimptotama hiperbole. Jednadžbe tih pravaca su:

$$y = \frac{b}{a} \cdot x \quad , \quad y = -\frac{b}{a} \cdot x.$$

1.inačica

$$\left. \begin{array}{l} y = 2 \cdot x \\ y = \frac{b}{a} \cdot x \end{array} \right\} \Rightarrow \frac{b}{a} = 2 \Rightarrow \frac{b}{a} = 2 \ / \cdot a \Rightarrow b = 2 \cdot a.$$

Budući da točka A(5, 8) pripada (leži) na hiperboli, uvrstit ćemo njezine koordinate u jednadžbu krivulje.

$$\left. \begin{array}{l} A(x, y) = A(5, 8) \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \end{array} \right\} \Rightarrow \frac{5^2}{a^2} - \frac{8^2}{b^2} = 1 \Rightarrow \frac{25}{a^2} - \frac{64}{b^2} = 1.$$

Iz sustava jednadžbi dobije se realna poluos a i imaginarna poluos b.

$$\left. \begin{array}{l} b = 2 \cdot a \\ \frac{25}{a^2} - \frac{64}{b^2} = 1 \end{array} \right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{supstitucije} \end{array} \right] \Rightarrow \frac{25}{a^2} - \frac{64}{(2 \cdot a)^2} = 1 \Rightarrow \frac{25}{a^2} - \frac{64}{4 \cdot a^2} = 1 \Rightarrow \frac{25}{a^2} - \frac{16}{a^2} = 1 \Rightarrow$$

$$\Rightarrow \frac{25-16}{a^2} = 1 \Rightarrow \frac{9}{a^2} = 1 \Rightarrow a^2 = 9.$$

Sada računamo b^2.

$$\left. \begin{array}{l} a^2 = 9 \\ b = 2 \cdot a \end{array} \right\} \Rightarrow \left. \begin{array}{l} a^2 = 9 \\ b = 2 \cdot a \ / ^2 \end{array} \right\} \Rightarrow \left. \begin{array}{l} a^2 = 9 \\ b^2 = 4 \cdot a^2 \end{array} \right\} \Rightarrow b^2 = 4 \cdot 9 \Rightarrow b^2 = 36.$$

Jednadžba hiperbole glasi:

$$\left.\begin{array}{l} a^2=9 \; , \; b^2=36 \\ \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \end{array}\right\} \Rightarrow \frac{x^2}{9}-\frac{y^2}{36}=1.$$

Odgovor je pod B.

2.inačica

Budući da točka A(5, 8) pripada hiperboli, uvrstit ćemo njezine koordinate u jednadžbu krivulje u svakom odgovoru A, B, C i D. Jednakost koja bude istinita je traženi odgovor.

Odgovor A.

$$\left.\begin{array}{l} A(x,\ y)=A(5,\ 8) \\ \dfrac{x^2}{36}-\dfrac{y^2}{9}=1 \end{array}\right\} \Rightarrow \frac{5^2}{36}-\frac{8^2}{9}=1 \Rightarrow \frac{25}{36}-\frac{64}{9}=1 \Rightarrow \frac{25-256}{36}=1 \Rightarrow -\frac{231}{36}=1 \ \text{nije točno}$$

Odgovor B.

$$\left.\begin{array}{l} A(x,\ y)=A(5,\ 8) \\ \dfrac{x^2}{9}-\dfrac{y^2}{36}=1 \end{array}\right\} \Rightarrow \frac{5^2}{9}-\frac{8^2}{36}=1 \Rightarrow \frac{25}{9}-\frac{64}{36}=1 \Rightarrow \frac{100-64}{36}=1 \Rightarrow \frac{36}{36}=1 \ \text{točno je}$$

Odgovor je pod B.

Dalje ne moramo provjeravati jer je samo jedan odgovor točan.

Vježba 099

Asimptota hiperbole je pravac y = 2 · x. Na hiperboli je točka (– 5, – 8). Jednadžba hiperbole je:

$$A.\ \frac{x^2}{36}-\frac{y^2}{9}=1 \quad , \quad B.\ \frac{x^2}{9}-\frac{y^2}{36}=1 \quad , \quad C.\ \frac{x^2}{6}-\frac{y^2}{3}=1 \quad , \quad D.\ \frac{x^2}{3}-\frac{y^2}{6}=1$$

Rezultat: Odgovor je pod B.

Zadatak 100 (Mario, gimnazija)

Zadana je kružnica $(x-1)^2+(y+3)^2=17$.

a) Točka A(2, y), y > 0 pripada kružnici. Odredite y.

b) Odredite jednadžbu tangente na kružnicu u točki A.

Rješenje 100

Ponovimo!

Ako je S(p, q) središte kružnice, a r njezin polumjer, tada središnja jednadžba kružnice glasi

$$(x-p)^2+(y-q)^2=r^2.$$

Jednadžba tangente kružnice $(x-p)^2+(y-q)^2=r^2$ s diralištem $D(x_0, y_0)$ glasi

$$(x_0-p)\cdot(x-p)+(y_0-q)\cdot(y-q)=r^2.$$

a)

Budući da točka A pripada kružnici, prvo ćemo izračunati ordinatu te točke. U jednadžbu kružnice uvrstimo x = 2 i dobijemo kvadratnu jednadžbu.

$$\left.\begin{array}{l} A(x,\ y)=A(2,\ y) \\ (x-1)^2+(y+3)^2=17 \end{array}\right\} \Rightarrow (2-1)^2+(y+3)^2=17 \Rightarrow 1^2+(y+3)^2=17 \Rightarrow$$

$$\Rightarrow 1+(y+3)^2=17 \Rightarrow (y+3)^2=17-1 \Rightarrow (y+3)^2=16 \Rightarrow (y+3)^2=16 \ /\sqrt{\ } \Rightarrow y+3=\pm\sqrt{16} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} y+3=4 \\ y+3=-4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=4-3 \\ y=-4-3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y_1=1 \\ y_2=-7 \end{array}\right\}.$$

Ordinata koja zadovoljava uvjet $y > 0$ je $y_1 = 1$. Dakle, A(2, 1).

b)

Jednadžba tangente glasi:

$$\left.\begin{array}{l} (x-1)^2+(y+3)^2=17 \\ A(x_0,\ y_0)=A(2,1)\ \ ,\ \ p=1\ \ ,\ \ q=-3\ \ ,\ \ r^2=17 \\ (x_0-p)\cdot(x-p)+(y_0-q)\cdot(y-q)=r^2 \end{array}\right\} \Rightarrow (2-1)\cdot(x-1)+(1+3)\cdot(y+3)=17 \Rightarrow$$

$$\Rightarrow 1\cdot(x-1)+4\cdot(y+3)=17 \Rightarrow x-1+4\cdot y+12=17 \Rightarrow x-1+4\cdot y+12-17=0 \Rightarrow x+4\cdot y-6=0.$$

Vježba 100

Zadana je kružnica $(x-1)^2+(y+3)^2=17$. Točka A(2, y), $y < 0$ pripada kružnici. Odredite y.

Rezultat: A(2, – 7).