Zadatak 141 (Kranjc, pomorska škola)

Pojednostavnite: $\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}:\left(\frac{1}{a^{-1}}-\frac{1}{b^{-1}}\right)^{-1}.$

Rješenje 141

Ponovimo!

$$a^{-n}=\frac{1}{a^{n}} \quad , \quad \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}.$$

$$\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}:\left(\frac{1}{a^{-1}}-\frac{1}{b^{-1}}\right)^{-1}=\left(\frac{b-a}{a\cdot b}\right)^{-1}:(a-b)^{-1}=\frac{a\cdot b}{b-a}:\frac{1}{a-b}=\frac{a\cdot b}{b-a}\cdot\frac{a-b}{1}=\frac{a\cdot b}{b-a}\cdot\frac{-(b-a)}{1}=-a\cdot b.$$

Vježba 141

Pojednostavnite: $\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}\cdot\left(\frac{1}{a^{-1}}-\frac{1}{b^{-1}}\right).$

Rezultat: $-a\cdot b.$

Zadatak 142 (Los-Habanos, gimnazija)

Pojednostavnite izraz: $\frac{\left(b^{-2}-a^{-2}\right)^{-1}}{\left(a^{-2}+b^{-2}\right)\cdot\left(a\cdot b^{-3}-a^{-3}\cdot b\right)^{-1}}.$

Rješenje 142

Ponovimo!

$$\text{Razlika kvadrata}: (x+y)\cdot(x-y)=x^{2}-y^{2}, \quad a^{-n}=\frac{1}{a^{n}} \quad , \quad \frac{1}{a^{-n}}=a^{n}.$$

1.inačica

$$\frac{\left(b^{-2}-a^{-2}\right)^{-1}}{\left(a^{-2}+b^{-2}\right)\cdot\left(a\cdot b^{-3}-a^{-3}\cdot b\right)^{-1}}=\frac{a\cdot b^{-3}-a^{-3}\cdot b}{\left(a^{-2}+b^{-2}\right)\cdot\left(b^{-2}-a^{-2}\right)}=\frac{\frac{a}{b^{3}}-\frac{b}{a^{3}}}{\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}\right)\cdot\left(\frac{1}{b^{2}}-\frac{1}{a^{2}}\right)}=$$

$$=\frac{\frac{a^{4}-b^{4}}{a^{3}\cdot b^{3}}}{\frac{b^{2}+a^{2}}{a^{2}\cdot b^{2}}\cdot\frac{a^{2}-b^{2}}{a^{2}\cdot b^{2}}}=\frac{\frac{a^{4}-b^{4}}{a^{3}\cdot b^{3}}}{\frac{\left(b^{2}+a^{2}\right)\cdot\left(a^{2}-b^{2}\right)}{a^{4}\cdot b^{4}}}=\frac{\frac{a^{4}-b^{4}}{a^{3}\cdot b^{3}}}{\frac{\left(a^{2}+b^{2}\right)\cdot\left(a^{2}-b^{2}\right)}{a^{4}\cdot b^{4}}}=\frac{\frac{a^{4}-b^{4}}{a^{3}\cdot b^{3}}}{\frac{a^{4}-b^{4}}{a^{4}\cdot b^{4}}}=\frac{a^{4}\cdot b^{4}}{a^{3}\cdot b^{3}}=a\cdot b.$$

2.inačica

$$\frac{\left(b^{-2}-a^{-2}\right)^{-1}}{\left(a^{-2}+b^{-2}\right)\cdot\left(a\cdot b^{-3}-a^{-3}\cdot b\right)^{-1}}=\frac{a\cdot b^{-3}-a^{-3}\cdot b}{\left(a^{-2}+b^{-2}\right)\cdot\left(b^{-2}-a^{-2}\right)}=\frac{a\cdot b\cdot\left(b^{-4}-a^{-4}\right)}{\left(b^{-2}+a^{-2}\right)\cdot\left(b^{-2}-a^{-2}\right)}=$$

$$=\frac{a\cdot b\cdot\left(b^{-4}-a^{-4}\right)}{b^{-4}-a^{-4}}=a\cdot b.$$

Vježba 142

Pojednostavnite izraz: $\frac{\left(a\cdot b^{-3}-a^{-3}\cdot b\right)\cdot\left(a^{-2}+b^{-2}\right)^{-1}}{b^{-2}-a^{-2}}.$

Rezultat: $a\cdot b.$

Zadatak 143 (Los-Habanos, gimnazija)

Pojednostavnite izraz: $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{a}{x\cdot y}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{2}{x\cdot y}-\dfrac{a^2}{x^2\cdot y^2}}\cdot(a+x+y).$

Rješenje 143

Ponovimo!

Kvadrat zbroja: $(a+b)^2=a^2+2\cdot a\cdot b+b^2$, Razlika kvadrata: $(x+y)\cdot(x-y)=x^2-y^2$.

$$\frac{\frac{1}{x}+\frac{1}{y}-\frac{a}{x\cdot y}}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{2}{x\cdot y}-\frac{a^2}{x^2\cdot y^2}}\cdot(a+x+y)=\frac{\frac{y+x-a}{x\cdot y}}{\frac{y^2+x^2+2\cdot x\cdot y-a^2}{x^2\cdot y^2}}\cdot(a+x+y)=$$

$$=\frac{\frac{y+x-a}{1}}{\frac{y^2+x^2+2\cdot x\cdot y-a^2}{x\cdot y}}\cdot(a+x+y)=\frac{\frac{y+x-a}{1}}{\frac{(y+x)^2-a^2}{x\cdot y}}\cdot(a+x+y)=\frac{x\cdot y\cdot(y+x-a)}{(y+x)^2-a^2}\cdot(a+x+y)=$$

$$=\frac{x\cdot y\cdot(y+x-a)}{(y+x-a)\cdot(y+x+a)}\cdot(a+x+y)=x\cdot y.$$

Vježba 143

Pojednostavnite izraz: $(a+x+y):\dfrac{\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{2}{x\cdot y}-\dfrac{a^2}{x^2\cdot y^2}}{\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{a}{x\cdot y}}.$

Rezultat: x · y.

Zadatak 144 (Sanela, maturantica gimnazije)

Reducirajte izraz: $\left(x^3-1\right)\cdot(x+1)^{-1}\cdot\left(x^3+1\right)\cdot\left(x^4+x^2+1\right)^{-1}.$

Rješenje 144

Ponovimo!

$a^{-n}=\dfrac{1}{a^n}$, $a^3-b^3=(a-b)\cdot\left(a^2+a\cdot b+b^2\right)$, $a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right)$

$a^2-b^2=(a-b)\cdot(a+b)$, $a^2+2\cdot a\cdot b+b^2=(a+b)^2$.

Najprije rastavimo trinom $x^4 + x^2 + 1$ na faktore:

$$x^4+x^2+1=x^4+2\cdot x^2-x^2+1=\left(x^4+2\cdot x^2+1\right)-x^2=\left(x^2+1\right)^2-x^2=$$

$$=\left(x^2+1-x\right)\cdot\left(x^2+1+x\right)=\left(x^2-x+1\right)\cdot\left(x^2+x+1\right).$$

Sada reduciramo algebarski izraz:

$$\left(x^3-1\right)\cdot(x+1)^{-1}\cdot\left(x^3+1\right)\cdot\left(x^4+x^2+1\right)^{-1}=\left(x^3-1\right)\cdot\frac{1}{x+1}\cdot\left(x^3+1\right)\cdot\frac{1}{x^4+x^2+1}=$$

$$=\frac{x^3-1}{1}\cdot\frac{1}{x+1}\cdot\frac{x^3+1}{1}\cdot\frac{1}{x^4+x^2+1}=\frac{\left(x^3-1\right)\cdot\left(x^3+1\right)}{(x+1)\cdot\left(x^4+x^2+1\right)}=\frac{(x-1)\cdot\left(x^2+x+1\right)\cdot(x+1)\cdot\left(x^2-x+1\right)}{(x+1)\cdot\left(x^2+x+1\right)\cdot\left(x^2-x+1\right)}=$$

$$=\frac{(x-1)\cdot\left(x^2+x+1\right)\cdot(x+1)\cdot\left(x^2-x+1\right)}{(x+1)\cdot\left(x^2+x+1\right)\cdot\left(x^2-x+1\right)}=x-1.$$

Vježba 144

Reducirajte izraz: $\left(x^3-1\right)\cdot\left(x^2-1\right)^{-1}\cdot\left(x^3+1\right)\cdot\left(x^4+x^2+1\right)^{-1}.$

Rezultat: 1.

Zadatak 145 (Sanela, maturantica gimnazije)

Reducirajte izraz: $\frac{x^3+y^3}{x+y}:\left(x^2-y^2\right)+\frac{2\cdot y}{x+y}-\frac{x\cdot y}{x^2-y^2}.$

Rješenje 145

Ponovimo!

$$a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right)\quad,\quad a^2-b^2=(a-b)\cdot(a+b)\quad,\quad \frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}\quad,\quad \frac{n}{1}=n.$$

Pojednostavljujemo algebarski izraz:

$$\frac{x^3+y^3}{x+y}:\left(x^2-y^2\right)+\frac{2\cdot y}{x+y}-\frac{x\cdot y}{x^2-y^2}=\frac{(x+y)\cdot\left(x^2-x\cdot y+y^2\right)}{x+y}\cdot\frac{1}{x^2-y^2}+\frac{2\cdot y}{x+y}-\frac{x\cdot y}{(x-y)\cdot(x+y)}=$$

$$=\frac{x^2-x\cdot y+y^2}{1}\cdot\frac{1}{(x-y)\cdot(x+y)}+\frac{2\cdot y}{x+y}-\frac{x\cdot y}{(x-y)\cdot(x+y)}=\frac{x^2-x\cdot y+y^2}{(x-y)\cdot(x+y)}+\frac{2\cdot y}{x+y}-\frac{x\cdot y}{(x-y)\cdot(x+y)}=$$

$$=\frac{x^2-x\cdot y+y^2+2\cdot y\cdot(x-y)-x\cdot y}{(x-y)\cdot(x+y)}=\frac{x^2-x\cdot y+y^2+2\cdot x\cdot y-2\cdot y^2-x\cdot y}{(x-y)\cdot(x+y)}=\frac{x^2-y^2}{x^2-y^2}=1.$$

Vježba 145

Reducirajte izraz: $\frac{x^3+y^3}{x+y}:\frac{x^2-x\cdot y+y^2}{x}.$

Rezultat: x.

Zadatak 146 (Tajanstvena, ekonomska škola)

Pojednostavnite: $\sqrt[3]{a^2\cdot b}\cdot\sqrt[4]{a^3\cdot b^5}\cdot\sqrt[6]{a^{-1}\cdot b^{-2}}.$

Rješenje 146

Ponovimo!

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{a\cdot b}\quad,\quad \left(a^n\right)^m=a^{n\cdot m}\quad,\quad a^n\cdot a^m=a^{n+m}\quad,\quad a^n\cdot b^n=(a\cdot b)^n.$$

Budući da je 12 najmanji zajednički višekratnik brojeva 3, 4 i 6 sve korijene svedemo na 12 – ti korijen:

$$\sqrt[3]{a^2\cdot b}\cdot\sqrt[4]{a^3\cdot b^5}\cdot\sqrt[6]{a^{-1}\cdot b^{-2}}=\sqrt[12]{\left(a^2\cdot b\right)^4}\cdot\sqrt[12]{\left(a^3\cdot b^5\right)^3}\cdot\sqrt[12]{\left(a^{-1}\cdot b^{-2}\right)^2}=$$

$$=\sqrt[12]{a^8\cdot b^4}\cdot\sqrt[12]{a^9\cdot b^{15}}\cdot\sqrt[12]{a^{-2}\cdot b^{-4}}=\sqrt[12]{a^8\cdot b^4\cdot a^9\cdot b^{15}\cdot a^{-2}\cdot b^{-4}}=\sqrt[12]{a^{15}\cdot b^{15}}=$$

$$=\sqrt[12]{\left(a^5\cdot b^5\right)^3}=\sqrt[12]{\left(a^5\cdot b^5\right)^3}=\sqrt[4]{a^5\cdot b^5}=\sqrt[4]{(a\cdot b)^5}=\sqrt[4]{(a\cdot b)^4\cdot(a\cdot b)}=$$

$$=\sqrt[4]{(a\cdot b)^4}\cdot\sqrt[4]{a\cdot b}=\sqrt[4]{(a\cdot b)^4}\cdot\sqrt[4]{a\cdot b}=a\cdot b\cdot\sqrt[4]{a\cdot b}.$$

Vježba 146

Pojednostavnite: $\sqrt[3]{a^2\cdot b}\cdot\sqrt[6]{a^{-1}\cdot b^{-2}}.$

Rezultat: $\sqrt{a}.$

Zadatak 147 (Tajanstvena, ekonomska škola)

Izračunajte: $\sqrt{125}-5\cdot\sqrt{45}+27\cdot\sqrt{20}.$

Rješenje 147

Ponovimo!

$$\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}\quad,\quad\sqrt{a^2}=a,\ a\ge 0.$$

Svaki korijen djelomično korjenujemo:

- $\sqrt{125}=\sqrt{25\cdot 5}=\sqrt{25}\cdot\sqrt{5}=\sqrt{5^2}\cdot\sqrt{5}=5\cdot\sqrt{5}$
- $\sqrt{45}=\sqrt{9\cdot 5}=\sqrt{9}\cdot\sqrt{5}=\sqrt{3^2}\cdot\sqrt{5}=3\cdot\sqrt{5}$
- $\sqrt{20}=\sqrt{4\cdot 5}=\sqrt{4}\cdot\sqrt{5}=\sqrt{2^2}\cdot\sqrt{5}=2\cdot\sqrt{5}.$

Sada je:

$$\sqrt{125}-5\cdot\sqrt{45}+27\cdot\sqrt{20}=\sqrt{25\cdot 5}-5\cdot\sqrt{9\cdot 5}+27\cdot\sqrt{4\cdot 5}=5\cdot\sqrt{5}-5\cdot 3\cdot\sqrt{5}+27\cdot 2\cdot\sqrt{5}=$$
$$=5\cdot\sqrt{5}-15\cdot\sqrt{5}+54\cdot\sqrt{5}=44\cdot\sqrt{5}.$$

Vježba 147

Izračunajte: $\sqrt{125}-5\cdot\sqrt{45}.$

Rezultat: $-10\cdot\sqrt{5}.$

Zadatak 148 (Matea, Dajana, Anamarija, hotelijerska škola)

Pojednostavnite: $4^{2.5}\cdot 64^{-\frac{7}{6}}\cdot\left(\frac{1}{8}\right)^{-\frac{2}{3}}.$

Rješenje 148

Ponovimo!

$$\left(a^n\right)^m=a^{n\cdot m}\quad,\quad a^{-n}=\frac{1}{a^n}\quad,\quad a^n\cdot a^m=a^{n+m}\quad,\quad a^0=1.$$

$$4^{2.5}\cdot 64^{-\frac{7}{6}}\cdot\left(\frac{1}{8}\right)^{-\frac{2}{3}}=\left(2^2\right)^{2.5}\cdot\left(2^6\right)^{-\frac{7}{6}}\cdot\left(\frac{1}{2^3}\right)^{-\frac{2}{3}}=2^5\cdot 2^{-7}\cdot\left(2^{-3}\right)^{-\frac{2}{3}}=2^5\cdot 2^{-7}\cdot 2^2=2^0=1.$$

Vježba 148

Pojednostavnite: $4^{2.5}\cdot 64^{\frac{7}{6}}.$

Rezultat: $2^{12}.$

Zadatak 149 (Ana, ekomomska škola)

Izračunajte: $\left(2\cdot a-\frac{4\cdot a-1}{2\cdot a}\right)\cdot\frac{4\cdot a^2}{1-4\cdot a^2}:\left(\frac{1}{a}-\frac{4}{1+2\cdot a}\right).$

Rješenje 149

Ponovimo!

$$x^2-y^2=(x-y)\cdot(x+y) \quad , \quad x^2-2\cdot x\cdot y+y^2=(x-y)^2.$$

$$\left(2\cdot a-\frac{4\cdot a-1}{2\cdot a}\right)\cdot\frac{4\cdot a^2}{1-4\cdot a^2}:\left(\frac{1}{a}-\frac{4}{1+2\cdot a}\right)=\frac{(2\cdot a)^2-(4\cdot a-1)}{2\cdot a}\cdot\frac{4\cdot a^2}{1-4\cdot a^2}:\frac{1+2\cdot a-4\cdot a}{a\cdot(1+2\cdot a)}=$$

$$=\frac{4\cdot a^2-4\cdot a+1}{2\cdot a}\cdot\frac{4\cdot a^2}{(1-2\cdot a)\cdot(1+2\cdot a)}:\frac{1-2\cdot a}{a\cdot(1+2\cdot a)}=\frac{1-4\cdot a+4\cdot a^2}{2\cdot a}\cdot\frac{4\cdot a^2}{(1-2\cdot a)\cdot(1+2\cdot a)}\cdot\frac{a\cdot(1+2\cdot a)}{1-2\cdot a}=$$

$$=\frac{(1-2\cdot a)^2}{2}\cdot\frac{4\cdot a^2}{1-2\cdot a}\cdot\frac{1}{1-2\cdot a}=\frac{(1-2\cdot a)^2}{2}\cdot\frac{4\cdot a^2}{(1-2\cdot a)^2}=\frac{4\cdot a^2}{2}=2\cdot a^2.$$

Vježba 149

Izračunajte: $\left(2\cdot a-\frac{4\cdot a-1}{2\cdot a}\right):\frac{1-4\cdot a^2}{4\cdot a^2}:\left(\frac{1}{a}-\frac{4}{1+2\cdot a}\right).$

Rezultat: $2\cdot a^2.$

Zadatak 150 (Ana, ekomomska škola)

Izračunajte: $\left\{a^4-\left(a^2-1\right)\cdot\left[(a-1)^2+2\cdot a\right]\right\}^2.$

Rješenje 150

Ponovimo!

$$(x-y)^2=x^2-2\cdot x\cdot y+y^2 \quad , \quad (x-y)\cdot(x+y)=x^2-y^2 \quad , \quad \left(a^n\right)^m=a^{n\cdot m}.$$

$$\left\{a^4-\left(a^2-1\right)\cdot\left[(a-1)^2+2\cdot a\right]\right\}^2=\left\{a^4-\left(a^2-1\right)\cdot\left[a^2-2\cdot a+1+2\cdot a\right]\right\}^2=\left\{a^4-\left(a^2-1\right)\cdot\left[a^2+1\right]\right\}^2=$$

$$=\left\{a^4-\left(\left(a^2\right)^2-1\right)\right\}^2=\left\{a^4-\left(a^4-1\right)\right\}^2=\left\{a^4-a^4+1\right\}^2=1^2=1.$$

Vježba 150

Izračunajte: $\left\{a^4-\left(a^2-1\right)\cdot\left[(a+1)^2-2\cdot a\right]\right\}^2.$

Rezultat: 1.

Zadatak 151 (Rea, gimnazija)

Skrati razlomak: $\frac{a^2+2\cdot a\cdot b+b^2-c^2}{a^2+c^2+a\cdot b+2\cdot a\cdot c+b\cdot c}.$

Rješenje 151

Ponovimo!

$$x^2+2\cdot x\cdot y+y^2=(x+y)^2 \quad , \quad x\cdot y+x\cdot z=x\cdot(y+z) \quad , \quad x^2-y^2=(x-y)\cdot(x+y).$$

$$\frac{a^2+2\cdot a\cdot b+b^2-c^2}{a^2+c^2+a\cdot b+2\cdot a\cdot c+b\cdot c}=\left[2\cdot a\cdot c=a\cdot c+a\cdot c\right]=\frac{(a+b)^2-c^2}{a^2+a\cdot b+a\cdot c+a\cdot c+b\cdot c+c^2}=$$

$$=\frac{(a+b-c)\cdot(a+b+c)}{a\cdot(a+b+c)+c\cdot(a+b+c)}=\frac{(a+b-c)\cdot(a+b+c)}{(a+b+c)\cdot(a+c)}=\frac{a+b-c}{a+c}.$$

Vježba 151

Skrati razlomak: $\frac{a^2+c^2+a\cdot b+2\cdot a\cdot c+b\cdot c}{a^2+2\cdot a\cdot b+b^2-c^2}.$

Rezultat: $\dfrac{a+c}{a+b-c}.$

Zadatak 152 (Jan, Zoran, Luka, gimnazija)

Pojednostavni: $$\left[\frac{1}{\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^{-2}}-\left(\frac{\sqrt{a}-\sqrt{b}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}}\right)^{-1}\right]\cdot(a\cdot b)^{-\frac{1}{2}}.$$

Rješenje 152

Ponovimo!

$$a^{-n}=\frac{1}{a^{n}}\quad,\quad\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}\quad,\quad\sqrt[n]{a^{m}}=a^{\frac{m}{n}}\quad,\quad(x+y)^{2}=x^{2}+2\cdot x\cdot y+y^{2}$$

$$x^{3}-y^{3}=(x-y)\cdot\left(x^{2}+x\cdot y+y^{2}\right)\quad,\quad\sqrt{a^{2}}=a\ ,\ a\geq 0\quad,\quad\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}.$$

$$\left[\frac{1}{\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^{-2}}-\left(\frac{\sqrt{a}-\sqrt{b}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}}\right)^{-1}\right]\cdot(a\cdot b)^{-\frac{1}{2}}=\left[\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^{2}-\frac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{\sqrt{a}-\sqrt{b}}\right]\cdot\frac{1}{(a\cdot b)^{\frac{1}{2}}}=$$

$$=\left[\left(\sqrt{a}+\sqrt{b}\right)^{2}-\frac{\left(\sqrt{a}\right)^{3}-\left(\sqrt{b}\right)^{3}}{\sqrt{a}-\sqrt{b}}\right]\cdot\frac{1}{\sqrt{a\cdot b}}=\left[a+2\cdot\sqrt{a\cdot b}+b-\frac{\left(\sqrt{a}-\sqrt{b}\right)\cdot\left(a+\sqrt{a\cdot b}+b\right)}{\sqrt{a}-\sqrt{b}}\right]\cdot\frac{1}{\sqrt{a\cdot b}}=$$

$$=\left[a+2\cdot\sqrt{a\cdot b}+b-\left(a+\sqrt{a\cdot b}+b\right)\right]\cdot\frac{1}{\sqrt{a\cdot b}}=\left[a+2\cdot\sqrt{a\cdot b}+b-a-\sqrt{a\cdot b}-b\right]\cdot\frac{1}{\sqrt{a\cdot b}}=\sqrt{a\cdot b}\cdot\frac{1}{\sqrt{a\cdot b}}=1.$$

Vježba 152

Pojednostavni: $$\left[\frac{1}{\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^{-2}}-\left(\frac{\sqrt{a}-\sqrt{b}}{a^{\frac{3}{2}}-b^{\frac{3}{2}}}\right)^{-1}\right].$$

Rezultat: $\sqrt{a\cdot b}.$

Zadatak 153 (Hrvoje, gimnazija)

Neka je $x=\dfrac{b^{2}+c^{2}-a^{2}}{2\cdot b\cdot c}$ i $y=\dfrac{(a-b+c)\cdot(a+b-c)}{(a+b+c)\cdot(b+c-a)}$. Izračunajte (x + 1) · (y + 1).

Rješenje 153

Ponovimo!

Kvadrat zbroja: $(a+b)^{2}=a^{2}+2\cdot a\cdot b+b^{2}$, razlika kvadrata: $(x+y)\cdot(x-y)=x^{2}-y^{2}$

kvadrat razlike: $(a-b)^{2}=a^{2}-2\cdot a\cdot b+b^{2}$.

Računamo:

- $x+1=\frac{b^2+c^2-a^2}{2\cdot b\cdot c}+1=\frac{b^2+c^2-a^2+2\cdot b\cdot c}{2\cdot b\cdot c}=\frac{-a^2+b^2+c^2+2\cdot b\cdot c}{2\cdot b\cdot c}.$
- $y+1=\frac{(a-b+c)\cdot(a+b-c)}{(a+b+c)\cdot(b+c-a)}+1=\frac{(a-b+c)\cdot(a+b-c)+(a+b+c)\cdot(b+c-a)}{(a+b+c)\cdot(b+c-a)}=$

$$=\frac{(a-(b-c))\cdot(a+(b-c))+((b+c)+a)\cdot((b+c)-a)}{((b+c)+a)\cdot((b+c)-a)}=\frac{a^2-(b-c)^2+(b+c)^2-a^2}{(b+c)^2-a^2}=$$

$$=\frac{-(b-c)^2+(b+c)^2}{(b+c)^2-a^2}=\frac{-\left(b^2-2\cdot b\cdot c+c^2\right)+b^2+2\cdot b\cdot c+c^2}{b^2+2\cdot b\cdot c+c^2-a^2}=\frac{-b^2+2\cdot b\cdot c-c^2+b^2+2\cdot b\cdot c+c^2}{-a^2+b^2+c^2+2\cdot b\cdot c}=$$

$$=\frac{4\cdot b\cdot c}{-a^2+b^2+c^2+2\cdot b\cdot c}.$$

Sada izravno slijedi:

$$(x+1)\cdot(y+1)=\frac{-a^2+b^2+c^2+2\cdot b\cdot c}{2\cdot b\cdot c}\cdot\frac{4\cdot b\cdot c}{-a^2+b^2+c^2+2\cdot b\cdot c}=2.$$

Vježba 153

Neka je $x=\frac{b^2+c^2-a^2}{2\cdot b\cdot c}$ i $y=\frac{(a-b+c)\cdot(a+b-c)}{(a+b+c)\cdot(b+c-a)}$. Izračunajte (2 · x + 2) · (y + 1).

Rezultat: 4.

Zadatak 154 (Marijana, maturantica)

Ako je $x^{-x}-x=1$, pojednostavnite $\left(1+\frac{1}{x}\right)^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)^{-\frac{1}{x}}$.

Rješenje 154

Ponovimo!

$$\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}\quad,\quad a^n\cdot b^n=(a\cdot b)^n\quad,\quad\left(a^n\right)^m=a^{n\cdot m}\quad,\quad a^{-n}=\frac{1}{a^n}.$$

$$x^{-x}-x=1\Rightarrow x^{-x}=1+x.$$

Sada je:

$$\left(1+\frac{1}{x}\right)^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)^{-\frac{1}{x}}=\left(1+\frac{1}{x}\right)^{\frac{1}{x}}\cdot x^{\frac{1}{x}}=\left(x\cdot\left(1+\frac{1}{x}\right)\right)^{\frac{1}{x}}=(x+1)^{\frac{1}{x}}=\left(x^{-x}\right)^{\frac{1}{x}}=x^{-1}=\frac{1}{x}.$$

Vježba 154

Ako je $x^{x}-x=1$, pojednostavnite $\left(1+\frac{1}{x}\right)^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)^{-\frac{1}{x}}$.

Rezultat: x.

Zadatak 155 (Vedran, gimnazija)

Kolika je vrijednost izraza $x^4+\frac{1}{x^4}$, ako je $x^2-3\cdot x+1=0$?

Rješenje 155

Ponovimo!

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2\quad,\quad\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}\quad,\quad\left(a^n\right)^m=a^{n\cdot m}.$$

$$x^2-3\cdot x+1=0\ /:x \Rightarrow \frac{x^2}{x}-\frac{3\cdot x}{x}+\frac{1}{x}=0 \Rightarrow x-3+\frac{1}{x}=0 \Rightarrow x+\frac{1}{x}=3\ /^2 \Rightarrow \left(x+\frac{1}{x}\right)^2=3^2 \Rightarrow$$

$$\Rightarrow x^2+2\cdot x\cdot\frac{1}{x}+\left(\frac{1}{x}\right)^2=9 \Rightarrow x^2+2+\frac{1}{x^2}=9 \Rightarrow x^2+\frac{1}{x^2}=9-2 \Rightarrow x^2+\frac{1}{x^2}=7\ /^2 \Rightarrow$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=7^2 \Rightarrow \left(x^2\right)^2+2\cdot x^2\cdot\frac{1}{x^2}+\left(\frac{1}{x^2}\right)^2=7^2 \Rightarrow x^4+2+\frac{1}{x^4}=49 \Rightarrow$$

$$\Rightarrow x^4+\frac{1}{x^4}=49-2 \Rightarrow x^4+\frac{1}{x^4}=47.$$

Vježba 155

Kolika je vrijednost izraza $x^4+\frac{1}{x^4}$, ako je $x^2-4\cdot x+1=0$?

Rezultat: 194.

Zadatak 156 (Iva, gimnazija)

Kolika je vrijednost izraza x^3+y^3, ako je $x+y=4$ i $x^2+y^2=10$?

Rješenje 156

Ponovimo!

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right)$$

$$(a+b)^3=a^3+3\cdot a^2\cdot b+3\cdot a\cdot b^2+b^3.$$

1.inačica

$$\left.\begin{array}{l} x+y=4 \\ x^2+y^2=10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y=4\ /^2 \\ x^2+y^2=10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2+2\cdot x\cdot y+y^2=16 \\ x^2+y^2=10 \end{array}\right\} \Rightarrow \left[\begin{array}{c} \text{oduzmemo} \\ \text{jednadžbe} \end{array}\right] \Rightarrow$$

$$\Rightarrow x^2+2\cdot x\cdot y+y^2-x^2-y^2=16-10 \Rightarrow 2\cdot x\cdot y=6\ /:2 \Rightarrow x\cdot y=3.$$

Vrijednost zadanog izraza iznosi:

$$\left.\begin{array}{l} x+y=4\ ,\ x^2+y^2=10\ ,\ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2-x\cdot y+y^2\right) \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y=4\ ,\ x^2+y^2=10\ ,\ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2+y^2-x\cdot y\right) \end{array}\right\} \Rightarrow$$

$$\Rightarrow x^3+y^3=4\cdot(10-3) \Rightarrow x^3+y^3=4\cdot 7 \Rightarrow x^3+y^3=28.$$

2.inačica

Zbog identiteta

$$x\cdot y=\frac{1}{2}\cdot\left[(x+y)^2-\left(x^2+y^2\right)\right],$$

vrijedi

$$x\cdot y=\frac{1}{2}\cdot\left[4^2-10\right] \Rightarrow x\cdot y=\frac{1}{2}\cdot[16-10] \Rightarrow x\cdot y=\frac{1}{2}\cdot 6 \Rightarrow x\cdot y=3.$$

Vrijednost zadanog izraza iznosi:

$$\left.\begin{array}{l} x+y=4\ ,\ x^2+y^2=10\ ,\ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2-x\cdot y+y^2\right) \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y=4\ ,\ x^2+y^2=10\ ,\ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2+y^2-x\cdot y\right) \end{array}\right\} \Rightarrow$$

$$\Rightarrow x^3+y^3=4\cdot(10-3) \Rightarrow x^3+y^3=4\cdot 7 \Rightarrow x^3+y^3=28.$$

3.inačica

$$\left.\begin{matrix} x+y=4 \\ x^2+y^2=10 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x+y=4 \ /^2 \\ x^2+y^2=10 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x^2+2\cdot x\cdot y+y^2=16 \\ x^2+y^2=10 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 2\cdot x\cdot y=16-x^2-y^2 \\ x^2+y^2=10 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} 2\cdot x\cdot y=16-\left(x^2+y^2\right) \\ x^2+y^2=10 \end{matrix}\right\} \Rightarrow 2\cdot x\cdot y=16-10 \Rightarrow 2\cdot x\cdot y=6 \ /:2 \Rightarrow x\cdot y=3.$$

Vrijednost zadanog izraza iznosi:

$$\left.\begin{matrix} x+y=4 \ , \ x^2+y^2=10 \ , \ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2-x\cdot y+y^2\right) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x+y=4 \ , \ x^2+y^2=10 \ , \ x\cdot y=3 \\ x^3+y^3=(x+y)\cdot\left(x^2+y^2-x\cdot y\right) \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow x^3+y^3=4\cdot(10-3) \Rightarrow x^3+y^3=4\cdot 7 \Rightarrow x^3+y^3=28.$$

Vježba 156

Kolika je vrijednost izraza x^3+y^3, ako je $x+y=5$ i $x^2+y^2=5$?

Rezultat: -25.

Zadatak 157 (Elena, gimnazija)

Riješite jednadžbu: $\frac{a}{b}\cdot\left(1-\frac{a}{x}\right)+\frac{b}{a}\cdot\left(1-\frac{b}{x}\right)=1.$

Rješenje 157

Ponovimo!

$$a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right) \quad , \quad \frac{a}{b}=\frac{c}{d} \Rightarrow \frac{b}{a}=\frac{d}{c}.$$

1.inačica

$$\frac{a}{b}\cdot\left(1-\frac{a}{x}\right)+\frac{b}{a}\cdot\left(1-\frac{b}{x}\right)=1 \Rightarrow \frac{a}{b}-\frac{a^2}{b\cdot x}+\frac{b}{a}-\frac{b^2}{a\cdot x}=1 \ /\cdot a\cdot b\cdot x \Rightarrow a^2\cdot x-a^3+b^2\cdot x-b^3=a\cdot b\cdot x \Rightarrow$$

$$\Rightarrow a^2\cdot x+b^2\cdot x-a\cdot b\cdot x=a^3+b^3 \Rightarrow \left(a^2+b^2-a\cdot b\right)\cdot x=a^3+b^3 \ /\cdot\frac{1}{a^2+b^2-a\cdot b} \Rightarrow$$

$$\Rightarrow x=\frac{a^3+b^3}{a^2+b^2-a\cdot b} \Rightarrow x=\frac{(a+b)\cdot\left(a^2-a\cdot b+b^2\right)}{a^2-a\cdot b+b^2} \Rightarrow x=a+b.$$

2.inačica

$$\frac{a}{b}\cdot\left(1-\frac{a}{x}\right)+\frac{b}{a}\cdot\left(1-\frac{b}{x}\right)=1 \Rightarrow \frac{a}{b}-\frac{a^2}{b\cdot x}+\frac{b}{a}-\frac{b^2}{a\cdot x}=1 \Rightarrow \frac{a}{b}+\frac{b}{a}-1=\frac{a^2}{b\cdot x}+\frac{b^2}{a\cdot x} \Rightarrow$$

$$\Rightarrow \frac{a^2+b^2-a\cdot b}{a\cdot b}=\frac{1}{x}\cdot\left(\frac{a^2}{b}+\frac{b^2}{a}\right) \Rightarrow \frac{a^2+b^2-a\cdot b}{a\cdot b}=\frac{1}{x}\cdot\frac{a^3+b^3}{a\cdot b} \ /\cdot\frac{a\cdot b\cdot x}{a^2+b^2-a\cdot b} \Rightarrow$$

$$\Rightarrow x=\frac{a^3+b^3}{a^2+b^2-a\cdot b} \Rightarrow x=\frac{(a+b)\cdot\left(a^2-a\cdot b+b^2\right)}{a^2-a\cdot b+b^2} \Rightarrow x=a+b.$$

Vježba 157

Riješite jednadžbu: $\frac{a}{b}\cdot\left(1-\frac{a}{x}\right)+\frac{b}{a}\cdot\left(1-\frac{b}{x}\right)=0.$

Rezultat: $x=\frac{a^3+b^3}{a^2+b^2}.$

Zadatak 158 (Ana, gimnazija)

Kolika je vrijednost broja x + y ako je $\left(x+\sqrt{x^2+1}\right)\cdot\left(y+\sqrt{y^2+1}\right)=1?$

Rješenje 158

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2.$$

Zadanu jednakost pomnožimo sa $x-\sqrt{x^2+1}$ i sa $y-\sqrt{y^2+1}$:

$$\left.\begin{array}{l}\left(x+\sqrt{x^2+1}\right)\cdot\left(y+\sqrt{y^2+1}\right)=1\ /\cdot\left(x-\sqrt{x^2+1}\right)\\ \left(x+\sqrt{x^2+1}\right)\cdot\left(y+\sqrt{y^2+1}\right)=1\ /\cdot\left(y-\sqrt{y^2+1}\right)\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}\left(x+\sqrt{x^2+1}\right)\cdot\left(x-\sqrt{x^2+1}\right)\cdot\left(y+\sqrt{y^2+1}\right)=x-\sqrt{x^2+1}\\ \left(x+\sqrt{x^2+1}\right)\cdot\left(y+\sqrt{y^2+1}\right)\cdot\left(y-\sqrt{y^2+1}\right)=y-\sqrt{y^2+1}\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}\left(x^2-\left(x^2+1\right)\right)\cdot\left(y+\sqrt{y^2+1}\right)=x-\sqrt{x^2+1}\\ \left(x+\sqrt{x^2+1}\right)\cdot\left(y^2-\left(y^2+1\right)\right)=y-\sqrt{y^2+1}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\left(x^2-x^2-1\right)\cdot\left(y+\sqrt{y^2+1}\right)=x-\sqrt{x^2+1}\\ \left(x+\sqrt{x^2+1}\right)\cdot\left(y^2-y^2-1\right)=y-\sqrt{y^2+1}\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}-1\cdot\left(y+\sqrt{y^2+1}\right)=x-\sqrt{x^2+1}\\ -1\cdot\left(x+\sqrt{x^2+1}\right)=y-\sqrt{y^2+1}\end{array}\right\}\Rightarrow\left.\begin{array}{l}-y-\sqrt{y^2+1}=x-\sqrt{x^2+1}\\ -x-\sqrt{x^2+1}=y-\sqrt{y^2+1}\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}-y-x=-\sqrt{x^2+1}+\sqrt{y^2+1}\ /\cdot(-1)\\ -x-y=-\sqrt{y^2+1}+\sqrt{x^2+1}\ /\cdot(-1)\end{array}\right\}\Rightarrow\left.\begin{array}{l}x+y=\sqrt{x^2+1}-\sqrt{y^2+1}\\ x+y=\sqrt{y^2+1}-\sqrt{x^2+1}\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{zbrojimo}\\ \text{jednadžbe}\end{array}\right]\Rightarrow$$

$$\Rightarrow 2\cdot(x+y)=\sqrt{x^2+1}-\sqrt{y^2+1}+\sqrt{y^2+1}-\sqrt{x^2+1}\Rightarrow 2\cdot(x+y)=0\ /:2\Rightarrow x+y=0.$$

Vježba 158

Kolika je vrijednost broja x + y ako je $\left(x-\sqrt{x^2+1}\right)\cdot\left(y-\sqrt{y^2+1}\right)=1?$

Rezultat: 0.

Zadatak 159 (Tina, gimnazija)

Dokažite da za svaki realni broj x vrijedi nejednakost: $x^2 - x + 2 > 0$.

Rješenje 159

Ponovimo!

$$a^2-2\cdot a\cdot b+b^2=(a-b)^2\quad,\quad a^2\ge 0\ ,\ a\in R.$$

$$x^2-x+2=x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+2=x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+2=x^2-x+\frac{1}{4}-\frac{1}{4}+2=$$

$$=\left(x^2-x+\frac{1}{4}\right)-\frac{1}{4}+2=\left(x-\frac{1}{2}\right)^2+\frac{7}{4}\geq\frac{7}{4}>0.$$

Dakle, za svaki realni broj x vrijedi:

$$\left(x-\frac{1}{2}\right)^2+\frac{7}{4}>0\ \Rightarrow\ x^2-x+2>0.$$

Vježba 159

Dokažite da za svaki realni broj x vrijedi nejednakost: $x^2 - x + 1 > 0$.

Rezultat: $\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4}>0.$

Zadatak 160 (Tina, gimnazija)

Dokažite da za sve realne brojeve x i y vrijedi nejednakost: $x^2 + y^2 + 1 \geq x \cdot y + x + y$.

Rješenje 160

Ponovimo!

$$a^2-2\cdot a\cdot b+b^2=(a-b)^2\quad,\quad a^2\geq 0\ ,\ a\in R.$$

$$x^2+y^2+1\geq x\cdot y+x+y\ \Rightarrow\ x^2+y^2+1-x\cdot y-x-y\geq 0\ /\cdot 2\ \Rightarrow\ 2\cdot x^2+2\cdot y^2+2-2\cdot x\cdot y-2\cdot x-2\cdot y\geq 0\ \Rightarrow$$

$$\Rightarrow\ x^2-2\cdot x\cdot y+y^2+x^2-2\cdot x+1+y^2-2\cdot y+1\geq 0\ \Rightarrow\ \left(x^2-2\cdot x\cdot y+y^2\right)+\left(x^2-2\cdot x+1\right)+\left(y^2-2\cdot y+1\right)\geq 0\ \Rightarrow$$

$$\Rightarrow\ (x-y)^2+(x-1)^2+(y-1)^2\geq 0.$$

Dobivena nejednakost istinita je za sve realne brojeve x i y pa je i polazna nejednakost istinita za sve x i y. Jednakost vrijedi ako i samo ako je x = y = 1.

Vježba 160

Dokažite da za sve realne brojeve x i y vrijedi nejednakost: $x^2 + y^2 \geq 2 \cdot (x + y - 1)$.

Rezultat: $(x-1)^2+(y-1)^2\geq 0.$