Zadatak 141 (Ana, gimnazija)

Sredi izraz: $\frac{1-\sin 2\alpha}{\cos 2\alpha}\cdot tg\left(\frac{\pi}{4}+\alpha\right).$

Rješenje 141

Ponovimo!

$$(a-b)^2=a^2-2\cdot a\cdot b+b^2 \quad , \quad a^2-b^2=(a-b)\cdot(a+b).$$

$$\cos^2 x+\sin^2 x=1 \quad , \quad \sin 2x=2\cdot\sin x\cdot\cos x \quad , \quad \cos 2x=\cos^2 x-\sin^2 x \quad , \quad tg\,(x+y)=\frac{tg\,x+tg\,y}{1-tg\,x\cdot tg\,y}.$$

Uporabom temeljnog identiteta, formula za sinus i kosinus dvostrukog kuta, formule za kvadrat razlike, formule za razliku kvadrata i adicijske formule za tangens dobije se:

$$\frac{1-\sin 2\alpha}{\cos 2\alpha}\cdot tg\left(\frac{\pi}{4}+\alpha\right)=\frac{\cos^2\alpha+\sin^2\alpha-2\cdot\sin\alpha\cdot\cos\alpha}{\cos^2\alpha-\sin^2\alpha}\cdot\frac{tg\,\frac{\pi}{4}+tg\,\alpha}{1-tg\,\frac{\pi}{4}\cdot tg\,\alpha}=$$

$$=\frac{\cos^2\alpha-2\cdot\cos\alpha\cdot\sin\alpha+\sin^2\alpha}{\cos^2\alpha-\sin^2\alpha}\cdot\frac{1+tg\,\alpha}{1-1\cdot tg\,\alpha}=\frac{(\cos\alpha-\sin\alpha)^2}{(\cos\alpha-\sin\alpha)\cdot(\cos\alpha+\sin\alpha)}\cdot\frac{1+tg\,\alpha}{1-tg\,\alpha}=$$

$$=\frac{(\cos\alpha-\sin\alpha)^2}{(\cos\alpha-\sin\alpha)\cdot(\cos\alpha+\sin\alpha)}\cdot\frac{1+\frac{\sin\alpha}{\cos\alpha}}{1-\frac{\sin\alpha}{\cos\alpha}}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\cdot\frac{\frac{\cos\alpha+\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}}=$$

$$=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\cdot\frac{\frac{\cos\alpha+\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\cdot\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\cdot\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=1.$$

Vježba 141

Sredi izraz: $\frac{\sin(x+y)}{\cos x\cdot\cos y}.$

Rezultat: $tg\,x+tg\,y.$

Zadatak 142 (Veronika, gimnazija)

Za koji $m\in R$ jednadžba $\sqrt{3}\cdot\sin x+\cos x=m$ ima realna rješenja?

Rješenje 142

Ponovimo!

$$tg\,\alpha=\frac{\sin\alpha}{\cos\alpha} \quad , \quad \cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta \quad , \quad |\cos\alpha|\le 1 \Leftrightarrow -1\le\cos\alpha\le 1 \text{ za svaki } \alpha.$$

$$\sqrt{3}\cdot\sin x+\cos x=m \Rightarrow tg\,\frac{\pi}{3}\cdot\sin x+\cos x=m \Rightarrow \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}\cdot\sin x+\cos x=m \Rightarrow$$

$$\Rightarrow \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}\cdot\sin x+\cos x=m \ /\cdot\cos\frac{\pi}{3} \Rightarrow \sin\frac{\pi}{3}\cdot\sin x+\cos x\cdot\cos\frac{\pi}{3}=m\cdot\cos\frac{\pi}{3} \Rightarrow$$

$$\Rightarrow \cos x \cdot \cos\frac{\pi}{3} + \sin x \cdot \sin\frac{\pi}{3} = m \cdot \frac{1}{2} \Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2} \cdot m \Rightarrow \left[-1 \le \cos\alpha \le 1\right] \Rightarrow$$

$$\Rightarrow -1 \le \frac{1}{2} \cdot m \le 1 \;/\cdot 2 \Rightarrow -2 \le m \le 2 \Rightarrow m \in \left[-2, 2\right].$$

Vježba 142

Za koji $m \in R$ jednadžba $\sin x + \sqrt{3} \cdot \cos x = m$ ima realna rješenja?

Rezultat: $m \in \left[-2, 2\right]$.

Zadatak 143 (Marija, srednja škola)

Izračunaj $3 \cdot tg\, x + 3 \cdot ctg\, x$ ako je $2 \cdot \sin x + 2 \cdot \cos x = 1$.

Rješenje 143

Ponovimo!

$$(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad \cos^2\alpha + \sin^2\alpha = 1 \quad , \quad tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} \quad , \quad ctg\,\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

$$2 \cdot \sin x + 2 \cdot \cos x = 1 \Rightarrow 2 \cdot \sin x + 2 \cdot \cos x = 1 \;/:2 \Rightarrow \sin x + \cos x = \frac{1}{2} \Rightarrow \left[\begin{array}{l}\text{kvadriramo} \\ \text{jednakost}\end{array}\right] \Rightarrow$$

$$\Rightarrow (\sin x + \cos x)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \sin^2 x + 2 \cdot \sin x \cdot \cos x + \cos^2 x = \frac{1}{4} \Rightarrow 1 + 2 \cdot \sin x \cdot \cos x = \frac{1}{4} \Rightarrow$$

$$\Rightarrow 2 \cdot \sin x \cdot \cos x = \frac{1}{4} - 1 \Rightarrow 2 \cdot \sin x \cdot \cos x = -\frac{3}{4} \;/:2 \Rightarrow \sin x \cdot \cos x = -\frac{3}{8}.$$

Sada je:

$$3 \cdot tg\, x + 3 \cdot ctg\, x = 3 \cdot (tg\, x + ctg\, x) = 3 \cdot \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) = 3 \cdot \frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x} = 3 \cdot \frac{1}{\sin x \cdot \cos x} =$$

$$= 3 \cdot \frac{1}{-\frac{3}{8}} = 3 \cdot \left(-\frac{8}{3}\right) = 3 \cdot \left(-\frac{8}{3}\right) = -8.$$

Vježba 143

Izračunaj $6 \cdot tg\, x + 6 \cdot ctg\, x$ ako je $2 \cdot \sin x + 2 \cdot \cos x = 1$.

Rezultat: – 16.

Zadatak 144 (Maturant, gimnazija)

Dokaži jednakost: $\frac{1}{\sin 10^0} - 4 \cdot \sin 70^0 = 2.$

Rješenje 144

Ponovimo!

$$\sin x \cdot \sin y = \frac{1}{2} \cdot \left[\cos(x-y) - \cos(x+y)\right] \quad , \quad \cos\left(90^0 - x\right) = \sin x.$$

$$\frac{1}{\sin 10^0} - 4 \cdot \sin 70^0 = \frac{1 - 4 \cdot \sin 70^0 \cdot \sin 10^0}{\sin 10^0} = \frac{1 - 4 \cdot \frac{1}{2} \cdot \left[\cos\left(70^0 - 10^0\right) - \cos\left(70^0 + 10^0\right)\right]}{\sin 10^0} =$$

$$= \frac{1 - 2 \cdot \left[\cos 60^0 - \cos 80^0\right]}{\sin 10^0} = \frac{1 - 2 \cdot \left[\frac{1}{2} - \cos 80^0\right]}{\sin 10^0} = \frac{1 - 1 + 2 \cdot \cos 80^0}{\sin 10^0} = \frac{1 - 1 + 2 \cdot \cos 80^0}{\sin 10^0} =$$

$$=\frac{2\cdot\cos 80^0}{\sin 10^0}=\frac{2\cdot\sin 10^0}{\sin 10^0}=\frac{2\cdot\sin 10^0}{\sin 10^0}=2.$$

Vježba 144

Dokaži jednakost: $\frac{\cos 80^0}{\sin 10^0}-\frac{\sin 25^0}{\cos 65^0}=0.$

Rezultat: Točna je.

Zadatak 145 (Melita, gimnazija)

Pojednostavni: $\frac{2\cdot\cos^2 x-1}{\sqrt{1-2\cdot\sin^2 x}}.$

Rješenje 145

Ponovimo!

$$\sqrt{a^2}=a\ ,\ a\ge 0\qquad ,\qquad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

Trigonometrijske funkcije polovičnog argumenta

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}\ \Rightarrow\ 2\cdot\sin^2\frac{\alpha}{2}=1-\cos\alpha\qquad ,\qquad \cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}\ \Rightarrow\ 2\cdot\cos^2\frac{\alpha}{2}=1+\cos\alpha.$$

$$\frac{2\cdot\cos^2 x-1}{\sqrt{1-2\cdot\sin^2 x}}=\frac{1+\cos 2x-1}{\sqrt{1-(1-\cos 2x)}}=\frac{1+\cos 2x-1}{\sqrt{1-1+\cos 2x}}=\frac{1+\cos 2x-1}{\sqrt{1-1+\cos 2x}}=\frac{\cos 2x}{\sqrt{\cos 2x}}=$$

$$=\frac{\sqrt{\cos^2 2x}}{\sqrt{\cos 2x}}=\sqrt{\frac{\cos^2 2x}{\cos 2x}}=\sqrt{\frac{\cos^2 2x}{\cos 2x}}=\sqrt{\cos 2x}.$$

Vježba 145

Pojednostavni: $\frac{\sqrt{1-2\cdot\sin^2 x}}{2\cdot\cos^2 x-1}.$

Rezultat: $\frac{1}{\sqrt{\cos 2x}}.$

Zadatak 146 (Ines, kemijska škola)

Dokaži jednakost: $\sin^4\alpha+\cos^2\alpha+\sin^2\alpha\cdot\cos^2\alpha=1.$

Rješenje 146

Ponovimo!

$$\sin^2 x+\cos^2 x=1.$$

$$\sin^4\alpha+\cos^2\alpha+\sin^2\alpha\cdot\cos^2\alpha=\sin^4\alpha+\sin^2\alpha\cdot\cos^2\alpha+\cos^2\alpha=\sin^2\alpha\cdot\left(\sin^2\alpha+\cos^2\alpha\right)+\cos^2\alpha=$$

$$=\sin^2\alpha\cdot 1+\cos^2\alpha=\sin^2\alpha+\cos^2\alpha=1.$$

Vježba 146

Dokaži jednakost: $\cos^4\alpha+\sin^2\alpha+\cos^2\alpha\cdot\sin^2\alpha=1.$

Rezultat: Dokaz analogan.

Zadatak 147 (Kiki, maturantica gimnazije)

Dokaži jednakost: $2\cdot\left(\sin^6\alpha+\cos^6\alpha\right)-3\cdot\left(\sin^4\alpha+\cos^4\alpha\right)+1=0.$

Rješenje 147

Ponovimo!

$$\cos^2\alpha+\sin^2\alpha=1\quad,\quad\sin^2\alpha=1-\cos^2\alpha\quad,\quad\cos^2\alpha=1-\sin^2\alpha.$$

$$a^4+2\cdot a^2\cdot b^2+b^4=\left(a^2+b^2\right)^2.$$

1.inačica

$$2\cdot\left(\sin^6\alpha+\cos^6\alpha\right)-3\cdot\left(\sin^4\alpha+\cos^4\alpha\right)+1=2\cdot\sin^6\alpha+2\cdot\cos^6\alpha-3\cdot\sin^4\alpha-3\cdot\cos^4\alpha+1=$$

$$=2\cdot\sin^6\alpha+2\cdot\cos^6\alpha-2\cdot\sin^4\alpha-\sin^4\alpha-2\cdot\cos^4\alpha-\cos^4\alpha+1=$$

$$=2\cdot\sin^6\alpha-2\cdot\sin^4\alpha+2\cdot\cos^6\alpha-2\cdot\cos^4\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha+2\cdot\sin^6\alpha-2\cdot\cos^4\alpha+2\cdot\cos^6\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha\left(1-\sin^2\alpha\right)-2\cdot\cos^4\alpha\left(1-\cos^2\alpha\right)-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha\cdot\cos^2\alpha-2\cdot\cos^4\alpha\cdot\sin^2\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha\cdot\left(\sin^2\alpha+\cos^2\alpha\right)-\sin^4\alpha-\cos^4\alpha+1=-2\cdot\sin^2\alpha\cdot\cos^2\alpha\cdot1-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha-\sin^4\alpha-\cos^4\alpha+1=-\left(2\cdot\sin^2\alpha\cdot\cos^2\alpha+\sin^4\alpha+\cos^4\alpha\right)+1=$$

$$=-\left(\sin^4\alpha+2\cdot\sin^2\alpha\cdot\cos^2\alpha+\cos^4\alpha\right)+1=-\left(\sin^2\alpha+\cos^2\alpha\right)^2+1=-1^2+1=-1+1=0$$

2.inačica

$$2\cdot\left(\sin^6\alpha+\cos^6\alpha\right)-3\cdot\left(\sin^4\alpha+\cos^4\alpha\right)+1=2\cdot\sin^6\alpha+2\cdot\cos^6\alpha-3\cdot\sin^4\alpha-3\cdot\cos^4\alpha+1=$$

$$=2\cdot\sin^6\alpha+2\cdot\cos^6\alpha-2\cdot\sin^4\alpha-\sin^4\alpha-2\cdot\cos^4\alpha-\cos^4\alpha+1=$$

$$=2\cdot\sin^6\alpha-2\cdot\sin^4\alpha+2\cdot\cos^6\alpha-2\cdot\cos^4\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha+2\cdot\sin^6\alpha-2\cdot\cos^4\alpha+2\cdot\cos^6\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha\left(1-\sin^2\alpha\right)-2\cdot\cos^4\alpha\left(1-\cos^2\alpha\right)-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^4\alpha\cdot\cos^2\alpha-2\cdot\cos^4\alpha\cdot\sin^2\alpha-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha\cdot\left(\sin^2\alpha+\cos^2\alpha\right)-\sin^4\alpha-\cos^4\alpha+1=-2\cdot\sin^2\alpha\cdot\cos^2\alpha\cdot1-\sin^4\alpha-\cos^4\alpha+1=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha-\sin^4\alpha-\cos^4\alpha+1=-2\cdot\sin^2\alpha\cdot\cos^2\alpha-\sin^4\alpha-\cos^4\alpha+\cos^2\alpha+\sin^2\alpha=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha+\sin^2\alpha-\sin^4\alpha+\cos^2\alpha-\cos^4\alpha=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha+\sin^2\alpha\cdot\left(1-\sin^2\alpha\right)+\cos^2\alpha\cdot\left(1-\cos^2\alpha\right)=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha+\sin^2\alpha\cdot\cos^2\alpha+\cos^2\alpha\cdot\sin^2\alpha=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha+\sin^2\alpha\cdot\cos^2\alpha+\sin^2\alpha\cdot\cos^2\alpha=$$

$$=-2\cdot\sin^2\alpha\cdot\cos^2\alpha+2\cdot\sin^2\alpha\cdot\cos^2\alpha=0.$$

Vježba 147

Dokaži jednakost: $\sin^6\alpha+\cos^6\alpha-\sin^4\alpha-\cos^4\alpha+\sin^2\alpha\cdot\cos^2\alpha=0.$

Rezultat: Dokaz analogan.

Zadatak 148 (Kiki, maturantica gimnazije)

Dokaži jednakost: $\dfrac{\sin^4\alpha+\cos^4\alpha-1}{\sin^6\alpha+\cos^6\alpha-1}=\dfrac{2}{3}.$

Rješenje 148

Ponovimo!

$$\cos^2x+\sin^2x=1\quad,\quad\sin^2x=1-\cos^2x\quad,\quad\cos^2x=1-\sin^2x.$$

$$a^2-b^2=(a-b)\cdot(a+b).$$

$$\frac{\sin^4\alpha+\cos^4\alpha-1}{\sin^6\alpha+\cos^6\alpha-1}=\frac{\sin^4\alpha+\cos^4\alpha-\left(\sin^2\alpha+\cos^2\alpha\right)}{\sin^6\alpha+\cos^6\alpha-\left(\sin^2\alpha+\cos^2\alpha\right)}=\frac{\sin^4\alpha+\cos^4\alpha-\sin^2\alpha-\cos^2\alpha}{\sin^6\alpha+\cos^6\alpha-\sin^2\alpha-\cos^2\alpha}=$$

$$=\frac{\sin^4\alpha-\sin^2\alpha+\cos^4\alpha-\cos^2\alpha}{\sin^6\alpha-\sin^2\alpha+\cos^6\alpha-\cos^2\alpha}=\frac{-\sin^2\alpha+\sin^4\alpha-\cos^2\alpha+\cos^4\alpha}{-\sin^2\alpha+\sin^6\alpha-\cos^2\alpha+\cos^6\alpha}=$$

$$=\frac{-\sin^2\alpha\cdot\left(1-\sin^2\alpha\right)-\cos^2\alpha\cdot\left(1-\cos^2\alpha\right)}{-\sin^2\alpha\cdot\left(1-\sin^4\alpha\right)-\cos^2\alpha\cdot\left(1-\cos^4\alpha\right)}=$$

$$=\frac{-\sin^2\alpha\cdot\cos^2\alpha-\cos^2\alpha\cdot\sin^2\alpha}{-\sin^2\alpha\cdot\left(1-\sin^2\alpha\right)\cdot\left(1+\sin^2\alpha\right)-\cos^2\alpha\cdot\left(1-\cos^2\alpha\right)\cdot\left(1+\cos^2\alpha\right)}=$$

$$=\frac{-2\cdot\sin^2\alpha\cdot\cos^2\alpha}{-\sin^2\alpha\cdot\cos^2\alpha\cdot\left(1+\sin^2\alpha\right)-\cos^2\alpha\cdot\sin^2\alpha\cdot\left(1+\cos^2\alpha\right)}=\frac{-2\cdot\sin^2\alpha\cdot\cos^2\alpha}{-\sin^2\alpha\cdot\cos^2\alpha\cdot\left(1+\sin^2\alpha+1+\cos^2\alpha\right)}=$$

$$=\frac{-2\cdot\sin^2\alpha\cdot\cos^2\alpha}{-\sin^2\alpha\cdot\cos^2\alpha\cdot\left(1+\sin^2\alpha+1+\cos^2\alpha\right)}=\frac{2}{1+1+\sin^2\alpha+\cos^2\alpha}=\frac{2}{1+1+1}=\frac{2}{3}.$$

Vježba 148

Dokaži jednakost: $\dfrac{\sin^6\alpha+\cos^6\alpha-1}{\sin^4\alpha+\cos^4\alpha-1}=\dfrac{3}{2}.$

Rezultat: Dokaz analogan.

Zadatak 149 (Kiki, maturantica gimnazije)

Dokaži jednakost: $\cos^2\alpha-\cos^4\alpha+\sin^4\alpha=\sin^2\alpha.$

Rješenje 149

Ponovimo!

$$\cos^2x+\sin^2x=1\quad,\quad\sin^2x=1-\cos^2x\quad,\quad\cos^2x=1-\sin^2x.$$

$$a^2-b^2=(a-b)\cdot(a+b).$$

1.inačica

$$\cos^2\alpha-\cos^4\alpha+\sin^4\alpha=\cos^2\alpha-\left(\cos^4\alpha-\sin^4\alpha\right)=\cos^2\alpha-\left(\cos^2\alpha-\sin^2\alpha\right)\cdot\left(\cos^2\alpha+\sin^2\alpha\right)=$$

$$=\cos^2\alpha-\left(\cos^2\alpha-\sin^2\alpha\right)\cdot 1=\cos^2\alpha-\left(\cos^2\alpha-\sin^2\alpha\right)=\cos^2\alpha-\cos^2\alpha+\sin^2\alpha=$$

$$=\cos^2\alpha-\cos^2\alpha+\sin^2\alpha=\sin^2\alpha.$$

2.inačica

$$\cos^2\alpha-\cos^4\alpha+\sin^4\alpha=\cos^2\alpha\cdot\left(1-\cos^2\alpha\right)+\sin^4\alpha=\cos^2\alpha\cdot\sin^2\alpha+\sin^4\alpha=$$

$$=\sin^2\alpha\cdot\left(\cos^2\alpha+\sin^2\alpha\right)=\sin^2\alpha\cdot 1=\sin^2\alpha.$$

Vježba 149

Dokaži jednakost: $\sin^2\alpha-\sin^4\alpha+\cos^4\alpha=\cos^2\alpha.$

Rezultat: Dokaz analogan.

Zadatak 150 (Ana, gimnazija)

Ako je $\alpha+\beta=\frac{\pi}{2},\ \alpha\neq\beta,$ koliko je $\frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}$?

Rješenje 150

Ponovimo!

$$tg\left(\frac{\pi}{2}-x\right)=ctg\,x\quad,\quad tg\,x\cdot ctg\,x=1.$$

Funkcija razlike

$$tg\,(x-y)=\frac{tg\,x-tg\,y}{1+tg\,x\cdot tg\,y}.$$

$$\alpha+\beta=\frac{\pi}{2}\ \Rightarrow\ \beta=\frac{\pi}{2}-\alpha.$$

Sada vrijedi:

$$tg\,(\alpha-\beta)=\frac{tg\,\alpha-tg\,\beta}{1+tg\,\alpha\cdot tg\,\beta}\ \Rightarrow\ tg\,(\alpha-\beta)=\frac{tg\,\alpha-tg\,\beta}{1+tg\,\alpha\cdot tg\,\beta}\ /\cdot\frac{1}{tg\,\alpha-tg\,\beta}\ \Rightarrow\ \frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}=\frac{1}{1+tg\,\alpha\cdot tg\,\beta}\ \Rightarrow$$

$$\Rightarrow\left[\beta=\frac{\pi}{2}-\alpha\right]\Rightarrow\ \frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}=\frac{1}{1+tg\,\alpha\cdot tg\left(\frac{\pi}{2}-\alpha\right)}\ \Rightarrow\ \frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}=\frac{1}{1+tg\,\alpha\cdot ctg\,\alpha}\ \Rightarrow$$

$$\Rightarrow\ \frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}=\frac{1}{1+1}\ \Rightarrow\ \frac{tg\,(\alpha-\beta)}{tg\,\alpha-tg\,\beta}=\frac{1}{2}.$$

Vježba 150

Ako je $\alpha+\beta=\frac{\pi}{2},\ \alpha\neq\beta,$ koliko je $\frac{tg\,\alpha-tg\,\beta}{tg\,(\alpha-\beta)}$?

Rezultat: 2.

Zadatak 151 (Luka, gimnazija)

Ako je $\cos^2\alpha+\cos^2\beta=a,$ nađi $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta).$

Rješenje 151

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2 \quad , \quad (a\cdot b)^n=a^n\cdot b^n.$$

$$\cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y \quad , \quad \cos(x-y)=\cos x\cdot\cos y+\sin x\cdot\sin y.$$

$$\cos^2 x+\sin^2 x=1 \quad , \quad \cos x\cdot\cos y=\frac{1}{2}\cdot\left[\cos(x+y)+\cos(x-y)\right] \quad , \quad \cos 2x=2\cdot\cos^2 x-1.$$

1.inačica

$$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=(\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta)\cdot(\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta)=$$

$$=(\cos\alpha\cdot\cos\beta)^2-(\sin\alpha\cdot\sin\beta)^2=\cos^2\alpha\cdot\cos^2\beta-\sin^2\alpha\cdot\sin^2\beta=$$

$$=\cos^2\alpha\cdot\cos^2\beta-\left(1-\cos^2\alpha\right)\cdot\left(1-\cos^2\beta\right)=$$

$$=\cos^2\alpha\cdot\cos^2\beta-\left(1-\cos^2\beta-\cos^2\alpha+\cos^2\alpha\cdot\cos^2\beta\right)=$$

$$=\cos^2\alpha\cdot\cos^2\beta-1+\cos^2\beta+\cos^2\alpha-\cos^2\alpha\cdot\cos^2\beta=$$

$$=\cos^2\alpha\cdot\cos^2\beta-1+\cos^2\beta+\cos^2\alpha-\cos^2\alpha\cdot\cos^2\beta=-1+\cos^2\beta+\cos^2\alpha=$$

$$=\cos^2\alpha+\cos^2\beta-1=\left[\cos^2\alpha+\cos^2\beta=a\right]=a-1.$$

2.inačica

$$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=\frac{1}{2}\cdot\left[\cos((\alpha+\beta)+(\alpha-\beta))+\cos((\alpha+\beta)-(\alpha-\beta))\right]=$$

$$=\frac{1}{2}\cdot\left[\cos(\alpha+\beta+\alpha-\beta)+\cos(\alpha+\beta-\alpha+\beta)\right]=\frac{1}{2}\cdot\left[\cos(\alpha+\beta+\alpha-\beta)+\cos(\alpha+\beta-\alpha+\beta)\right]=$$

$$=\frac{1}{2}\cdot[\cos 2\alpha+\cos 2\beta]=\frac{1}{2}\cdot\left[2\cdot\cos^2\alpha-1+2\cdot\cos^2\beta-1\right]=\frac{1}{2}\cdot\left[2\cdot\cos^2\alpha+2\cdot\cos^2\beta-2\right]=$$

$$=\frac{1}{2}\cdot 2\cdot\left[\cos^2\alpha+\cos^2\beta-1\right]=\frac{1}{2}\cdot 2\cdot\left[\cos^2\alpha+\cos^2\beta-1\right]=\cos^2\alpha+\cos^2\beta-1=$$

$$=\left[\cos^2\alpha+\cos^2\beta=a\right]=a-1.$$

Vježba 151

Ako je $\cos^2\alpha+\cos^2\beta=1$, nađi $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)$.

Rezultat: 0.

Zadatak 152 (3A, TUPŠ)

Napiši 39° 8' 23" u radijanima.

Rješenje 152

Ponovimo!

Razmjer ili proporcija je jednakost dvaju jednakih omjera. Ako je

$$a : b = k \quad i \quad c : d = k,$$

tada je razmjer ili proporcija

$$a : b = c : d.$$

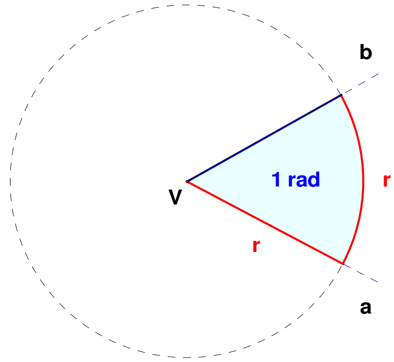
Umnožak vanjskih članova razmjera a i d jednak je umnošku unutarnjih članova razmjera b i c.

$$a : b = c : d \;\Rightarrow\; a\cdot d=b\cdot c.$$

Jedinica mjerenja kuta je 1° (stupanj), a njegovi dijelovi su 1' (minuta) i 1" (sekunda). Vrijedi:

$$1^0=60' \;\Rightarrow\; 1'=\left(\frac{1}{60}\right)^0 \quad , \quad 1'=60'' \;\Rightarrow\; 1''=\left(\frac{1}{60}\right)' \quad , \quad 1^0=3600'' \;\Rightarrow\; 1''=\left(\frac{1}{3600}\right)^0$$

Jedinica mjerenja kuta je i 1 rad (radijan). Središnji kut kružnice kojem je duljina pridruženog luka jednaka duljini polumjera kružnice je kut veličine 1 rad.



Kako jednu mjeru kuta pretvoriti u drugu?
Označimo s α° mjeru kuta u stupnjevima, a s α_r mjeru istog kuta u radijanima. Tada je

$\alpha_r : \alpha^0 = \pi : 180^0$	
$\alpha_r = \frac{\pi}{180^0} \cdot \alpha^0$	$\alpha^0 = \frac{180^0}{\pi} \cdot \alpha_r$

1.inačica
Najprije zadani kut pretvorimo u stupnjeve:

$$\alpha = 39^0 8' 23'' = 39^0 + \left(\frac{8}{60}\right)^0 + \left(\frac{23}{3600}\right)^0 = 39^0 + 0.13333333^0 + 0.00638889^0 = 39.13972222^0,$$

a tek tada pretvaramo ga u radijane:

$$\left.\begin{aligned} \alpha &= 39.13972222^0 \\ \alpha_r &= \frac{\pi}{180^0} \cdot \alpha \end{aligned}\right\} \Rightarrow \alpha_r = \frac{\pi}{180^0} \cdot 39.13972222^0 \Rightarrow \alpha_r = 0.68311702.$$

2.inačica
Najprije zadani kut pretvorimo u stupnjeve:

$$\alpha = 39^0 8' 23'' = 39^0 + \left(\frac{8}{60}\right)^0 + \left(\frac{23}{3600}\right)^0 = 39^0 + 0.13333333^0 + 0.00638889^0 = 39.13972222^0,$$

a tek tada u radijane uporabom razmjera:

$$\alpha_r : \alpha = \pi : 180^0 \Rightarrow 180^0 \cdot \alpha_r = \alpha \cdot \pi \; /:180^0 \Rightarrow \alpha_r = \frac{\alpha \cdot \pi}{180^0} \Rightarrow$$

$$\Rightarrow \alpha_r = \frac{39.13972222^0 \cdot \pi}{180^0} \Rightarrow \alpha_r = 0.68311702.$$

3.inačica
Najprije zadani kut pretvorimo u stupnjeve:

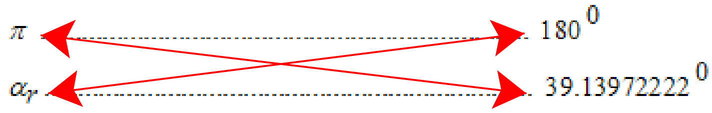
$$\alpha = 39^0 8' 23'' = 39^0 + \left(\frac{8}{60}\right)^0 + \left(\frac{23}{3600}\right)^0 = 39^0 + 0.13333333^0 + 0.00638889^0 = 39.13972222^0,$$

a tek tada u radijane pomoću pravila trojnog:

$$\pi \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 180^0$$

$$\alpha_r \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 39.13972222^0.$$

U prvi red napišemo da je π radijana isto što i 180°. U drugi red traženi broj radijana α_r potpisujemo pod π radijana, a 39.13972222° potpisujemo pod 180°. Sada u križ (dijagonalno) pomnožimo veličine i riješimo dobivenu jednadžbu po α_r:



$$180^0 \cdot \alpha_r = \pi \cdot 39.13972222^0 \Rightarrow 180^0 \cdot \alpha_r = \pi \cdot 39.13972222^0 \quad / : 180^0 \Rightarrow$$

$$\Rightarrow \alpha_r = \frac{\pi \cdot 39.13972222^0}{180^0} \Rightarrow \alpha_r = 0.68311702.$$

Vježba 152

Napiši 35° 12' 30" u radijanima.

Rezultat: 0.61450134.

Zadatak 153 (Ivana, kemijska škola)

Ako je sin x = a – cos x, nađi sin 2x.

Rješenje 153

Ponovimo!

$$(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad \cos^2 \alpha + \sin^2 \alpha = 1 \quad , \quad \sin 2\alpha = 2 \cdot \sin \alpha \cdot \cos \alpha.$$

$$\sin x = a - \cos x \Rightarrow \sin x + \cos x = a \Rightarrow \sin x + \cos x = a \quad /^2 \Rightarrow (\sin x + \cos x)^2 = a^2 \Rightarrow$$

$$\Rightarrow \sin^2 x + 2 \cdot \sin x \cdot \cos x + \cos^2 x = a^2 \Rightarrow \sin^2 x + \cos^2 x + 2 \cdot \sin x \cdot \cos x = a^2 \Rightarrow$$

$$\Rightarrow \sin^2 x + \cos^2 x + 2 \cdot \sin x \cdot \cos x = a^2 \Rightarrow 1 + 2 \cdot \sin x \cdot \cos x = a^2 \Rightarrow 1 + \sin 2x = a^2 \Rightarrow$$

$$\Rightarrow \sin 2x = a^2 - 1.$$

Vježba 153

Ako je sin x = 1 – cos x, nađi sin 2x.

Rezultat: 0.

Zadatak 154 (Iva, gimnazija)

Ako je tg x = 4, koliko je sin 4x?

Rješenje 154

Ponovimo!

$$tg\, x = \frac{\sin x}{\cos x} \Rightarrow \sin x = tg\, x \cdot \cos x \quad , \quad \left(a^n\right)^m = a^{n \cdot m} \quad , \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

$$a^n \cdot a^m = a^{n+m} \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2 \quad , \quad \left(\sqrt{a}\right)^4 = a^2.$$

Formule za izražavanje funkcija kosinus i sinus pomoću funkcije tangens istog kuta.

$$\cos \alpha = \frac{1}{\sqrt{1 + tg^2 \alpha}} \Rightarrow \cos^4 \alpha = \frac{1}{\left(1 + tg^2 \alpha\right)^2} \quad , \quad \sin \alpha = \frac{tg\, \alpha}{\sqrt{1 + tg^2 \alpha}}.$$

Funkcije dvostrukog kuta

$$\sin 2\alpha = 2 \cdot \sin \alpha \cdot \cos \alpha \quad , \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \quad , \quad tg\, 2\alpha = \frac{2 \cdot tg\, \alpha}{1 - tg^2 \alpha}.$$

1.inačica

Transformiramo sin 4x pomoću funkcija sinus i kosinus dvostrukog kuta.

$$\sin 4x = 2\cdot\sin 2x\cdot\cos 2x = 2\cdot 2\cdot\sin x\cdot\cos x\cdot\left(\cos^2 x-\sin^2 x\right) = 4\cdot\sin x\cdot\cos x\cdot\left(\cos^2 x-\sin^2 x\right) =$$
$$= 4\cdot\sin x\cdot\cos^3 x - 4\cdot\sin^3 x\cdot\cos x.$$

Iz uvjeta tg x = 4 dobije se

$$\left.\begin{array}{l} tg\ x = 4 \\ \sin x = tg\ x\cdot\cos x \end{array}\right\} \Rightarrow \sin x = 4\cdot\cos x.$$

Dalje slijedi

$$\left.\begin{array}{l} \sin 4x = 4\cdot\sin x\cdot\cos^3 x - 4\cdot\sin^3 x\cdot\cos x \\ \sin x = 4\cdot\cos x \end{array}\right\} \Rightarrow \left[\begin{array}{c}\text{metoda}\\ \text{supstitucije}\end{array}\right] \Rightarrow$$

$$\Rightarrow \sin 4x = 4\cdot 4\cdot\cos x\cdot\cos^3 x - 4\cdot\left(4\cdot\cos x\right)^3\cdot\cos x \Rightarrow \sin 4x = 16\cdot\cos^4 x - 4\cdot 64\cdot\cos^3 x\cdot\cos x \Rightarrow$$
$$\Rightarrow \sin 4x = 16\cdot\cos^4 x - 256\cdot\cos^4 x \Rightarrow \sin 4x = -240\cdot\cos^4 x.$$

Uporabom formule za izražavanje funkcije kosinus pomoću funkcije tangens istog kuta dobije se

$$\left.\begin{array}{l} tg\ x = 4\ ,\ \cos^4 x = \dfrac{1}{\left(1+tg^2 x\right)^2} \\ \sin 4x = -240\cdot\cos^4 x \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos^4 x = \dfrac{1}{\left(1+4^2\right)^2} \\ \sin 4x = -240\cdot\cos^4 x \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos^4 x = \dfrac{1}{\left(1+16\right)^2} \\ \sin 4x = -240\cdot\cos^4 x \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \cos^4 x = \dfrac{1}{17^2} \\ \sin 4x = -240\cdot\cos^4 x \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos^4 x = \dfrac{1}{289} \\ \sin 4x = -240\cdot\cos^4 x \end{array}\right\} \Rightarrow \sin 4x = -240\cdot\frac{1}{289} \Rightarrow \sin 4x = -\frac{240}{289}.$$

2.inačica

Transformiramo tg 4x pomoću funkcije tangens dvostrukog kuta.

$$tg\ 4x = \frac{2\cdot tg\ 2x}{1-tg^2\ 2x} \Rightarrow tg\ 4x = \frac{2\cdot\dfrac{2\cdot tg\ x}{1-tg^2 x}}{1-\left(\dfrac{2\cdot tg\ x}{1-tg^2 x}\right)^2} \Rightarrow tg\ 4x = \frac{\dfrac{4\cdot tg\ x}{1-tg^2 x}}{1-\dfrac{\left(2\cdot tg\ x\right)^2}{\left(1-tg^2 x\right)^2}} \Rightarrow$$

$$\Rightarrow tg\ 4x = \frac{\dfrac{4\cdot tg\ x}{1-tg^2 x}}{1-\dfrac{4\cdot tg^2 x}{\left(1-tg^2 x\right)^2}} \Rightarrow tg\ 4x = \frac{\dfrac{4\cdot tg\ x}{1-tg^2 x}}{\dfrac{\left(1-tg^2 x\right)^2 - 4\cdot tg^2 x}{\left(1-tg^2 x\right)^2}} \Rightarrow tg\ 4x = \frac{\dfrac{4\cdot tg\ x}{1-tg^2 x}}{\dfrac{\left(1-tg^2 x\right)^2 - 4\cdot tg^2 x}{\left(1-tg^2 x\right)^2}} \Rightarrow$$

$$\Rightarrow tg\ 4x = \frac{\dfrac{4\cdot tg\ x}{1}}{\dfrac{\left(1-tg^2 x\right)^2 - 4\cdot tg^2 x}{1-tg^2 x}} \Rightarrow tg\ 4x = \frac{4\cdot tg\ x\cdot\left(1-tg^2 x\right)}{\left(1-tg^2 x\right)^2 - 4\cdot tg^2 x} \Rightarrow$$

$$\Rightarrow tg\ 4x = \frac{4\cdot tg\ x\cdot\left(1-tg^2 x\right)}{1-2\cdot tg^2 x+tg^4 x - 4\cdot tg^2 x} \Rightarrow tg\ 4x = \frac{4\cdot tg\ x\cdot\left(1-tg^2 x\right)}{tg^4 x-6\cdot tg^2 x+1} \Rightarrow \left[\begin{array}{c}\text{uvjet}\\ tg\ x = 4\end{array}\right] \Rightarrow$$

$$\Rightarrow \ tg\, 4x=\frac{4\cdot 4\cdot\left(1-4^2\right)}{4^4-6\cdot 4^2+1} \ \Rightarrow \ tg\, 4x=\frac{16\cdot(-15)}{256-96+1} \ \Rightarrow \ tg\, 4x=-\frac{240}{161}.$$

Sada sin 4x iznosi:

$$\sin 4x=\frac{tg\, 4x}{\sqrt{1+tg^2\, 4x}} \ \Rightarrow \ \sin 4x=\frac{-\frac{240}{161}}{\sqrt{1+\left(-\frac{240}{161}\right)^2}} \ \Rightarrow \ \sin 4x=\frac{-\frac{240}{161}}{\sqrt{1+\frac{240^2}{161^2}}} \ \Rightarrow$$

$$\Rightarrow \ \sin 4x=\frac{-\frac{240}{161}}{\sqrt{\frac{161^2+240^2}{161^2}}} \ \Rightarrow \ \sin 4x=\frac{-\frac{240}{161}}{\frac{\sqrt{161^2+240^2}}{161}} \ \Rightarrow \ \sin 4x=\frac{-\frac{240}{161}}{\frac{\sqrt{83521}}{161}} \ \Rightarrow$$

$$\Rightarrow \ \sin 4x=-\frac{240}{\sqrt{83521}} \ \Rightarrow \ \sin 4x=-\frac{240}{289}.$$

Vježba 154

Ako je tg x = 1, koliko je sin 4x?

Rezultat: 0.

Zadatak 155 (Viki, gimnazija)

Koliko rješenja ima jednadžba $\sin^2 x+\sin^2\frac{x}{2}+\sin^2\frac{x}{3}+\ \ldots\ +\sin^2\frac{x}{100}=0$?

A. 100 *B.* 1 *C.* Beskonačno mnogo *D.* 0 *E.* 50

Rješenje 155

Ponovimo!

$$x^2+y^2=0 \ \Rightarrow \ x=y=0.$$

Ako je zbroj kvadrata realnih brojeva jednak nuli, tada je svaki realan broj nula.
Broj v višekratnik je cijelog broja b ako postoji cijeli broj n takav da vrijedi

$$v=n\cdot b.$$

$$\sin^2 x+\sin^2\frac{x}{2}+\sin^2\frac{x}{3}+\ \ldots\ +\sin^2\frac{x}{100}=0 \ \Rightarrow \ \left.\begin{matrix}\sin x=0\\ \sin\frac{x}{2}=0\\ \ldots\ldots\\ \sin\frac{x}{100}=0\end{matrix}\right\} \ \Rightarrow \ \left.\begin{matrix}x=k\cdot\pi \ , \ k\in Z\\ \frac{x}{2}=k\cdot\pi \ , \ k\in Z\\ \ldots\ldots\\ \frac{x}{100}=k\cdot\pi \ , \ k\in Z\end{matrix}\right\} \ \Rightarrow$$

$$\Rightarrow \ \left.\begin{matrix}x=k\cdot\pi \ , \ k\in Z\\ \frac{x}{2}=k\cdot\pi \ /\cdot 2 \ , \ k\in Z\\ \ldots\ldots\\ \frac{x}{100}=k\cdot\pi \ /\cdot 100 \ , \ k\in Z\end{matrix}\right\} \ \Rightarrow \ \left.\begin{matrix}x=k\cdot\pi \ , \ k\in Z\\ x=2\cdot k\cdot\pi \ , \ k\in Z\\ \ldots\ldots\\ x=100\cdot k\cdot\pi \ , \ k\in Z\end{matrix}\right\}.$$

Budući da svaki broj ima beskonačno mnogo višekratnika, zajedničkih višekratnika brojeva

$$k\cdot\pi \ , \ 2\cdot k\cdot\pi \ , \ \ldots \ , \ 100\cdot k\cdot\pi$$

bit će, također, beskonačno mnogo. Odgovor je pod C.

Vježba 155

Koliko rješenja ima jednadžba $\sin^2 x + \sin^2 \frac{x}{2} + \sin^2 \frac{x}{3} + \ \ldots \ + \sin^2 \frac{x}{10} = 0$?

$A.$ 100 $\quad B.$ 1 $\quad C.$ Beskonačno mnogo $\quad D.$ 0 $\quad E.$ 50

Rezultat: C.

Zadatak 156 (Dijana, srednja škola)

Pojednostavni: $\frac{1-\sin 2\alpha}{\cos 2\alpha} \cdot tg\left(\frac{\pi}{4}+\alpha\right)$.

Rješenje 156

Ponovimo!

$$\cos^2 x + \sin^2 x = 1 \quad , \quad \sin 2x = 2 \cdot \sin x \cdot \cos x \quad , \quad \cos 2x = \cos^2 x - \sin^2 x.$$

$$tg(x+y) = \frac{tg\, x + tg\, y}{1 - tg\, x \cdot tg\, y} \quad , \quad x^2 - y^2 = (x-y)\cdot(x+y) \quad , \quad (x-y)^2 = x^2 - 2\cdot x\cdot y + y^2.$$

$$\frac{1-\sin 2\alpha}{\cos 2\alpha} \cdot tg\left(\frac{\pi}{4}+\alpha\right) = \frac{\cos^2\alpha + \sin^2\alpha - 2\cdot\sin\alpha\cdot\cos\alpha}{\cos^2\alpha - \sin^2\alpha} \cdot \frac{tg\frac{\pi}{4} + tg\,\alpha}{1 - tg\frac{\pi}{4}\cdot tg\,\alpha} =$$

$$= \frac{\cos^2\alpha - 2\cdot\sin\alpha\cdot\cos\alpha + \sin^2\alpha}{\cos^2\alpha - \sin^2\alpha} \cdot \frac{1+tg\,\alpha}{1-1\cdot tg\,\alpha} = \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha-\sin\alpha)\cdot(\cos\alpha+\sin\alpha)} \cdot \frac{1+tg\,\alpha}{1-tg\,\alpha} =$$

$$= \frac{\cos^2\alpha - 2\cdot\sin\alpha\cdot\cos\alpha + \sin^2\alpha}{\cos^2\alpha - \sin^2\alpha} \cdot \frac{1+tg\,\alpha}{1-1\cdot tg\,\alpha} = \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha-\sin\alpha)\cdot(\cos\alpha+\sin\alpha)} \cdot \frac{1+tg\,\alpha}{1-tg\,\alpha} =$$

$$= \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha-\sin\alpha)\cdot(\cos\alpha+\sin\alpha)} \cdot \frac{1+\frac{\sin\alpha}{\cos\alpha}}{1-\frac{\sin\alpha}{\cos\alpha}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} \cdot \frac{\frac{\cos\alpha+\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}} =$$

$$= \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} \cdot \frac{\frac{\cos\alpha+\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} \cdot \frac{\frac{\cos\alpha+\sin\alpha}{1}}{\frac{\cos\alpha-\sin\alpha}{1}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} \cdot \frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha} = 1.$$

Vježba 156

Pojednostavni: $\frac{1-\sin 2\alpha}{\cos 2\alpha}$.

Rezultat: $\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$.

Zadatak 157 (Amela, studentica)

Riješi jednadžbu: $-2\cdot\cos\left(x - 180^0\right) - 1 = 0.$

Rješenje 157

Ponovimo!

Parnost kosinusa

Za svaki realni broj x vrijedi

$$\cos(-x) = \cos x,$$

tj. kosinus je parna funkcija.

Periodičnost kosinusa

Kosinus je periodična funkcija s temeljnim periodom 2 · π ili 360°.

$$\cos(\alpha + k \cdot 2 \cdot \pi) = \cos\alpha, \ k \in Z \quad , \quad \cos\left(\alpha + k \cdot 360^0\right) = \cos\alpha, \ k \in Z.$$

Skup rješenja jednadžbe

$$\cos x = |a| \ , \ |a| \le 1,$$

je

$$\left\{\pm x_0 + k \cdot 360^0 : k \in Z\right\},$$

gdje je

$$x_0 \in R$$

jedno rješenje te jednadžbe.

Zapamti!

$$\pi \text{ rad} = 180^\circ$$

$$-2 \cdot \cos\left(x - 180^0\right) - 1 = 0 \Rightarrow -2 \cdot \cos\left(x - 180^0\right) = 1 \Rightarrow -2 \cdot \cos\left(x - 180^0\right) = 1 \ /: (-2) \Rightarrow$$

$$\Rightarrow \cos\left(x - 180^0\right) = -\frac{1}{2}.$$

Uz supstituciju

$$t = x - 180^0$$

jednadžba poprima oblik

$$\cos t = -\frac{1}{2}.$$

Rješenja će biti oni realni brojevi x za koje točka E(x) na trigonometrijskoj kružnici ima apscisu $-\frac{1}{2}$.

Takvih točaka ima dvije. Za jednu od njih je

$$t_1 = \frac{2 \cdot \pi}{3} \text{ ili } t_1 = 120^0.$$

Kako je kosinus parna funkcija i broj

$$t_2 = -\frac{2 \cdot \pi}{3} \text{ ili } t_2 = -120^0$$

zadovoljava jednadžbu

$$\cos t = -\frac{1}{2}.$$

Zbog periodičnosti kosinusa i brojevi

$$t_1 = 120^0 + k \cdot 360^0 \quad , \quad t_2 = -120^0 + k \cdot 360^0$$

su rješenja jednadžbe.

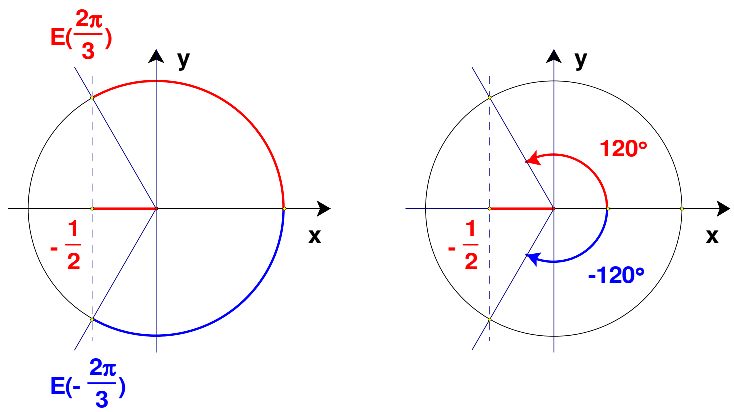
Sada se vraćamo na supstituciju.

- $$\left.\begin{array}{l} t = x - 180^0 \\ t = 120^0 + k \cdot 360^0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} t_1 = x_1 - 180^0 \\ t_1 = 120^0 + k \cdot 360^0 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{komparacije} \end{array}\right] \Rightarrow$$

$$\Rightarrow x_1 - 180^0 = 120^0 + k \cdot 360^0 \Rightarrow x_1 = 180^0 + 120^0 + k \cdot 360^0 \Rightarrow x_1 = 300^0 + k \cdot 360^0 \quad , \; k \in Z.$$

- $$\left.\begin{array}{l} t = x - 180^0 \\ t = -120^0 + k \cdot 360^0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} t_2 = x_2 - 180^0 \\ t_2 = -120^0 + k \cdot 360^0 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{komparacije} \end{array}\right] \Rightarrow$$

$$\Rightarrow x_2 - 180^0 = -120^0 + k \cdot 360^0 \Rightarrow x_2 = 180^0 - 120^0 + k \cdot 360^0 \Rightarrow x_2 = 60^0 + k \cdot 360^0 \quad , \; k \in Z.$$



Vježba 157

Riješi jednadžbu: $2 \cdot \cos\left(x - 180^0\right) + 1 = 0.$

Rezultat: $x_1 = 300^0 + k \cdot 360^0 \quad , \quad x_2 = 60^0 + k \cdot 360^0 \quad , \; k \in Z.$

Zadatak 158 (Maturant, gimnazija)

Ako jednadžba $\sin^4 x + \cos^4 x = a$ ima realna rješenja, tada a pripada intervalu:

$A. \left[0, \frac{1}{2}\right] \qquad B. \left\langle \frac{1}{2}, \frac{3}{2} \right\rangle \qquad C. \left\langle 1, \frac{3}{2} \right\rangle \qquad D. \left[\frac{1}{2}, 1\right]$

Rješenje 158

Ponovimo!

$$(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad , \quad (a \cdot b)^n = a^n \cdot b^n \quad , \quad \sin 2\alpha = 2 \cdot \sin\alpha \cdot \cos\alpha.$$

$$a \le b \Rightarrow a + c \le b + c \quad , \quad a \le b \Rightarrow a \cdot c \ge b \cdot c \; , \; c < 0 \quad , \quad \cos^2\alpha + \sin^2\alpha = 1.$$

Područje vrijednosti (kodomena) funkcije f(x) = sin x je segment $[-1, 1]$.

$$-1 \le \sin x \le 1 \Rightarrow 0 \le \sin^2 x \le 1.$$

$$\sin^4 x + \cos^4 x = \sin^4 x + 2 \cdot \sin^2 x \cdot \cos^2 x + \cos^4 x - 2 \cdot \sin^2 x \cdot \cos^2 x =$$

$$= \left(\sin^4 x + 2 \cdot \sin^2 x \cdot \cos^2 x + \cos^4 x\right) - 2 \cdot \sin^2 x \cdot \cos^2 x = \left(\sin^2 x + \cos^2 x\right)^2 - 2 \cdot \sin^2 x \cdot \cos^2 x =$$

$$= 1^2 - 2 \cdot \sin^2 x \cdot \cos^2 x = 1 - 2 \cdot \sin^2 x \cdot \cos^2 x = 1 - \frac{1}{2} \cdot 4 \cdot \sin^2 x \cdot \cos^2 x = 1 - \frac{1}{2} \cdot (2 \cdot \sin x \cdot \cos x)^2 =$$

$$= 1 - \frac{1}{2} \cdot (\sin 2x)^2 = 1 - \frac{1}{2} \cdot \sin^2 2x.$$

Iz uvjeta zadatka

$$\sin^4 x + \cos^4 x = a,$$

slijedi

$$1-\frac{1}{2}\cdot\sin^2 2x=a.$$

Budući da je

$$0\le\sin^2 2x\le 1,$$

dalje vrijedi:

$$0\le\sin^2 2x\le 1\ \Rightarrow\ 0\le\sin^2 2x\le 1\ /\cdot\left(-\frac{1}{2}\right)\ \Rightarrow\ 0\ge-\frac{1}{2}\cdot\sin^2 2x\ge-\frac{1}{2}\ \Rightarrow\ -\frac{1}{2}\le-\frac{1}{2}\cdot\sin^2 2x\le 0\ \Rightarrow$$

$$\Rightarrow\ -\frac{1}{2}\le-\frac{1}{2}\cdot\sin^2 2x\le 0\ /+1\ \Rightarrow\ -\frac{1}{2}+1\le-\frac{1}{2}\cdot\sin^2 2x+1\le 0+1\ \Rightarrow\ \frac{1}{2}\le 1-\frac{1}{2}\cdot\sin^2 2x\le 1\ \Rightarrow$$

$$\Rightarrow\ \frac{1}{2}\le a\le 1\ \Rightarrow\ a\in\left[\frac{1}{2},\ 1\right].$$

Odgovor je pod D.

Vježba 158

Ako jednadžba $\sin^4 x+\cos^4 x=\frac{a}{2}$ ima realna rješenja, tada a pripada intervalu:

A. $[1,\ 2]$ *B.* $\left\langle\frac{1}{2},\ 1\right\rangle$ *C.* $\left\langle 1,\ \frac{3}{2}\right\rangle$ *D.* $\left[\frac{1}{2},\ 2\right]$

Rezultat: A.

Zadatak 159 (Maturant, gimnazija)

U kojem se intervalu nalaze sve vrijednosti funkcije f(x) = a · sin x + b · cos x + c?

Rješenje 159

Ponovimo!

$$tg\,\alpha=\frac{\sin\alpha}{\cos\alpha}\quad,\quad\cos\alpha=\frac{1}{\sqrt{1+tg^2\alpha}}\quad,\quad\left(\frac{x}{y}\right)^n=\frac{x^n}{y^n}\quad,\quad\sqrt{a^2}=a\ ,\ a\ge 0.$$

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\sin\beta\cdot\cos\alpha\quad,\quad\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

$$a\le x\le b\ \Rightarrow\ a+c\le x+c\le b+c\quad,\quad a\le x\le b\ \Rightarrow\ a\cdot c\le c\cdot x\le b\cdot c\ ,\ c>0.$$

Područje vrijednosti (kodomena) funkcije f(x) = sin x je segment $[-1,\ 1]$.

$$-1\le\sin x\le 1\ \Rightarrow\ 0\le\sin^2 x\le 1.$$

$$f(x)=a\cdot\sin x+b\cdot\cos x+c\ \Rightarrow\ f(x)=a\cdot\left(\sin x+\frac{b}{a}\cdot\cos x\right)+c.$$

Da bismo izraz transformirali uvodimo zamjenu (supstituciju)

$$\frac{b}{a}=tg\,\varphi\ ,\ \varphi\in\left\langle 0,\ \frac{\pi}{2}\right\rangle.$$

Budući da je

$$\cos\varphi=\frac{1}{\sqrt{1+tg^2\varphi}},$$

slijedi

$$\cos\varphi=\frac{1}{\sqrt{1+tg^2\varphi}}\Rightarrow\cos\varphi=\frac{1}{\sqrt{1+\left(\frac{b}{a}\right)^2}}\Rightarrow\cos\varphi=\frac{1}{\sqrt{1+\frac{b^2}{a^2}}}\Rightarrow\cos\varphi=\frac{1}{\sqrt{\frac{a^2+b^2}{a^2}}}\Rightarrow$$

$$\Rightarrow\cos\varphi=\frac{1}{\frac{\sqrt{a^2+b^2}}{\sqrt{a^2}}}\Rightarrow\cos\varphi=\frac{1}{\frac{\sqrt{a^2+b^2}}{a}}\Rightarrow\cos\varphi=\frac{a}{\sqrt{a^2+b^2}}.$$

Sada je:

$$\left.\begin{array}{l} f(x)=a\cdot\left(\sin x+\frac{b}{a}\cdot\cos x\right)+c \\ \frac{b}{a}=tg\,\varphi \end{array}\right\}\Rightarrow f(x)=a\cdot(\sin x+tg\,\varphi\cdot\cos x)+c\Rightarrow$$

$$\Rightarrow f(x)=a\cdot\left(\sin x+\frac{\sin\varphi}{\cos\varphi}\cdot\cos x\right)+c\Rightarrow f(x)=\frac{a}{\cos\varphi}\cdot(\sin x\cdot\cos\varphi+\cos x\cdot\sin\varphi)+c\Rightarrow$$

$$\Rightarrow f(x)=\frac{a}{\cos\varphi}\cdot\sin(x+\varphi)+c\Rightarrow\left[\cos\varphi=\frac{a}{\sqrt{a^2+b^2}}\right]\Rightarrow f(x)=\frac{a}{\frac{a}{\sqrt{a^2+b^2}}}\cdot\sin(x+\varphi)+c\Rightarrow$$

$$\Rightarrow f(x)=\frac{\frac{a}{1}}{\frac{a}{\sqrt{a^2+b^2}}}\cdot\sin(x+\varphi)+c\Rightarrow f(x)=\frac{\frac{a}{1}}{\frac{a}{\sqrt{a^2+b^2}}}\cdot\sin(x+\varphi)+c\Rightarrow$$

$$\Rightarrow f(x)=\sqrt{a^2+b^2}\cdot\sin(x+\varphi)+c.$$

Odavde se dobije:

$$-1\le\sin(x+\varphi)\le1\Rightarrow-1\le\sin(x+\varphi)\le1\ /\cdot\sqrt{a^2+b^2}\Rightarrow$$

$$\Rightarrow-\sqrt{a^2+b^2}\le\sqrt{a^2+b^2}\cdot\sin(x+\varphi)\le\sqrt{a^2+b^2}\Rightarrow$$

$$\Rightarrow-\sqrt{a^2+b^2}\le\sqrt{a^2+b^2}\cdot\sin(x+\varphi)\le\sqrt{a^2+b^2}\ /+c\Rightarrow$$

$$\Rightarrow-\sqrt{a^2+b^2}+c\le\sqrt{a^2+b^2}\cdot\sin(x+\varphi)+c\le\sqrt{a^2+b^2}+c\Rightarrow$$

$$\Rightarrow-\sqrt{a^2+b^2}+c\le f(x)\le\sqrt{a^2+b^2}+c.$$

Vježba 159

U kojem se intervalu nalaze sve vrijednosti funkcije f(x) = a · sin x + b · cos x?

Rezultat: $-\sqrt{a^2+b^2}\le f(x)\le\sqrt{a^2+b^2}.$

Zadatak 160 (Ana, gimnazija)

Odredi vrijednost izraza $\frac{\sin 3x}{\sin x}-\frac{\cos 3x}{\cos x}$.

Rješenje 160

Ponovimo!

$$\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\sin\beta\cdot\cos\alpha \quad , \quad \sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha.$$

$$\frac{\sin 3x}{\sin x}-\frac{\cos 3x}{\cos x}=\frac{\sin 3x\cdot\cos x-\sin x\cdot\cos 3x}{\sin x\cdot\cos x}=\frac{\sin(3x-x)}{\sin x\cdot\cos x}=\frac{\sin 2x}{\sin x\cdot\cos x}=$$

$$=\frac{2\cdot\sin x\cdot\cos x}{\sin x\cdot\cos x}=\frac{2\cdot\sin x\cdot\cos x}{\sin x\cdot\cos x}=2.$$

Vježba 160

Odredi vrijednost izraza $\frac{\cos 3x}{\cos x}-\frac{\sin 3x}{\sin x}$.

Rezultat: – 2.