Zadatak 121 (Goran, gimnazija)

Odredi skup rješenja jednadžbe $\sin\left(\frac{x-\pi}{4}\right)\cdot\cos\left(\frac{x-\pi}{4}\right)=\frac{\sqrt{2}}{4}$ na segmentu $[0, 6\cdot\pi]$.

Rješenje 121

Ponovimo!

$$\sin 2\alpha = 2\cdot\sin\alpha\cdot\cos\alpha \quad , \quad a<b<c \Rightarrow a+x<b+x<c+x.$$

$$\sin\left(\frac{x-\pi}{4}\right)\cdot\cos\left(\frac{x-\pi}{4}\right)=\frac{\sqrt{2}}{4} \;/\cdot 2 \Rightarrow 2\cdot\sin\left(\frac{x-\pi}{4}\right)\cdot\cos\left(\frac{x-\pi}{4}\right)=\frac{\sqrt{2}}{2} \Rightarrow \sin 2\cdot\frac{x-\pi}{4}=\frac{\sqrt{2}}{2} \Rightarrow$$

$$\Rightarrow \sin\frac{x-\pi}{2}=\frac{\sqrt{2}}{2} \Rightarrow \left[\begin{array}{c}\text{supstitucija}\\ \frac{x-\pi}{2}=t\end{array}\right] \Rightarrow \sin t=\frac{\sqrt{2}}{2} \Rightarrow \left.\begin{array}{l} t_1=\frac{\pi}{4}+k\cdot 2\cdot\pi,\ k\in Z \\ t_2=\pi-\frac{\pi}{4}+k\cdot 2\cdot\pi,\ k\in Z\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} t_1=\frac{\pi}{4}+k\cdot 2\cdot\pi \\ t_2=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi\end{array}\right\}.$$

Vraćamo se supstituciji:

- $$\left.\begin{array}{l} t=\frac{\pi}{4}+k\cdot 2\cdot\pi \\ \frac{x-\pi}{2}=t\end{array}\right\} \Rightarrow \frac{x-\pi}{2}=\frac{\pi}{4}+k\cdot 2\cdot\pi \;/\cdot 2 \Rightarrow x-\pi=\frac{\pi}{2}+k\cdot 4\cdot\pi \Rightarrow$$

$$\Rightarrow x=\frac{\pi}{2}+\pi+k\cdot 4\cdot\pi \Rightarrow x=\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi.$$

- $$\left.\begin{array}{l} t=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi \\ \frac{x-\pi}{2}=t\end{array}\right\} \Rightarrow \frac{x-\pi}{2}=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi \;/\cdot 2 \Rightarrow x-\pi=\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi \Rightarrow$$

$$\Rightarrow x=\frac{3\cdot\pi}{2}+\pi+k\cdot 4\cdot\pi \Rightarrow x=\frac{5\cdot\pi}{2}+k\cdot 4\cdot\pi.$$

Budući da se rješenja moraju nalaziti na segmentu $[0, 6\cdot\pi]$, slijedi:

- $$\left.\begin{array}{l} 0\le x\le 6\cdot\pi \\ x=\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi\end{array}\right\} \Rightarrow 0\le\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi\le 6\cdot\pi \;/\cdot\frac{2}{\pi} \Rightarrow 0\le 3+8\cdot k\le 12 \;/-3 \Rightarrow$$

$$\Rightarrow 0-3\le 3+8\cdot k-3\le 12-3 \Rightarrow -3\le 8\cdot k\le 9 \;/:8 \Rightarrow \frac{-3}{8}\le k\le\frac{9}{8} \Rightarrow k=0, 1.$$

Rješenja su:

$$\left.\begin{array}{l} k=0 \\ x=\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi\end{array}\right\} \Rightarrow x_1=\frac{3\cdot\pi}{2}+0\cdot 4\cdot\pi \Rightarrow x_1=\frac{3\cdot\pi}{2},$$

$$\left.\begin{array}{l} k=1 \\ x=\frac{3\cdot\pi}{2}+k\cdot 4\cdot\pi\end{array}\right\} \Rightarrow x_2=\frac{3\cdot\pi}{2}+1\cdot 4\cdot\pi \Rightarrow x_2=\frac{3\cdot\pi}{2}+4\cdot\pi \Rightarrow x_2=\frac{11\cdot\pi}{2}.$$

- $$\left.\begin{array}{l} 0 \le x \le 6\cdot\pi \\ x=\frac{5\cdot\pi}{2}+k\cdot 4\cdot\pi \end{array}\right\} \Rightarrow 0 \le \frac{5\cdot\pi}{2}+k\cdot 4\cdot\pi \le 6\cdot\pi \ /\cdot\frac{2}{\pi} \Rightarrow 0 \le 5+8\cdot k \le 12 \ /-5 \Rightarrow$$

$$\Rightarrow 0-5 \le 5+8\cdot k-5 \le 12-5 \Rightarrow -5 \le 8\cdot k \le 7 \ /:8 \Rightarrow \frac{-5}{8} \le k \le \frac{7}{8} \Rightarrow k=0.$$

Rješenje je:

$$\left.\begin{array}{l} k=0 \\ x=\frac{5\cdot\pi}{2}+k\cdot 4\cdot\pi \end{array}\right\} \Rightarrow x_3=\frac{5\cdot\pi}{2}+0\cdot 4\cdot\pi \Rightarrow x_3=\frac{5\cdot\pi}{2}.$$

Vježba 121

Odredi skup rješenja jednadžbe $\sin\left(\frac{x-\pi}{4}\right)\cdot\cos\left(\frac{x-\pi}{4}\right)=\frac{\sqrt{2}}{4}$ na segmentu $[0,\ 2\cdot\pi]$.

Rezultat: $x=\frac{3\cdot\pi}{2}.$

Zadatak 122 (Vlado, srednja škola)

Koliko je $\frac{tg\,\alpha+ctg\,\alpha}{tg\,\alpha-ctg\,\alpha}$, ako je $\cos\alpha=\frac{1}{2}$?

Rješenje 122

Ponovimo!

$$tg\,\alpha=\frac{\sin\alpha}{\cos\alpha} \quad , \quad ctg\,\alpha=\frac{\cos\alpha}{\sin\alpha} \quad , \quad \cos^2\alpha+\sin^2\alpha=1.$$

$$\frac{tg\,\alpha+ctg\,\alpha}{tg\,\alpha-ctg\,\alpha}=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}}{\frac{\sin\alpha}{\cos\alpha}-\frac{\cos\alpha}{\sin\alpha}}=\frac{\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha\cdot\sin\alpha}}{\frac{\sin^2\alpha-\cos^2\alpha}{\cos\alpha\cdot\sin\alpha}}=\frac{\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha\cdot\sin\alpha}}{\frac{\sin^2\alpha-\cos^2\alpha}{\cos\alpha\cdot\sin\alpha}}=\frac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha-\cos^2\alpha}=\frac{1}{\sin^2\alpha-\cos^2\alpha}=$$

$$=\frac{1}{1-\cos^2\alpha-\cos^2\alpha}=\frac{1}{1-2\cdot\cos^2\alpha}=\frac{1}{1-2\cdot\left(\frac{1}{2}\right)^2}=\frac{1}{1-2\cdot\frac{1}{4}}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$$

Vježba 122

Koliko je $\frac{tg\,\alpha-ctg\,\alpha}{tg\,\alpha+ctg\,\alpha}$, ako je $\cos\alpha=\frac{1}{2}$?

Rezultat: $\frac{1}{2}.$

Zadatak 123 (Mornar, pomorska škola)

Ako je $\sin\frac{x}{2}+\cos\frac{x}{2}=\sqrt{2}$, nađi sin x.

Rješenje 123

Ponovimo!

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad \cos^2\alpha+\sin^2\alpha=1 \quad , \quad \sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha \quad , \quad \sin\alpha=2\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}.$$

$$\sin\frac{x}{2}+\cos\frac{x}{2}=\sqrt{2} \Rightarrow \left[\begin{array}{l}\text{kvadriramo}\\ \text{jednakost}\end{array}\right] \Rightarrow \sin\frac{x}{2}+\cos\frac{x}{2}=\sqrt{2}\ /^2 \Rightarrow \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2=\left(\sqrt{2}\right)^2 \Rightarrow$$

$$\Rightarrow \sin^2\frac{x}{2}+2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}+\cos^2\frac{x}{2}=2 \Rightarrow \left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}=2 \Rightarrow$$

$$\Rightarrow 1+\sin x=2 \Rightarrow \sin x=1.$$

Vježba 123

Ako je $\sin\frac{x}{2}+\cos\frac{x}{2}=1$, nađi sin x.

Rezultat: 0.

Zadatak 124 (Goga, srednja škola)

Mate i Sanja udaljeni su 2000 m i oboje gledaju balon. U isto vrijeme, Sanja vidi balon pod kutom 42º, a Mate pod kutom 70º. Koliko je Sanja udaljena od balona?

Rješenje 124

Ponovimo!

Zbroj unutarnjih kutova trokuta je 180º:

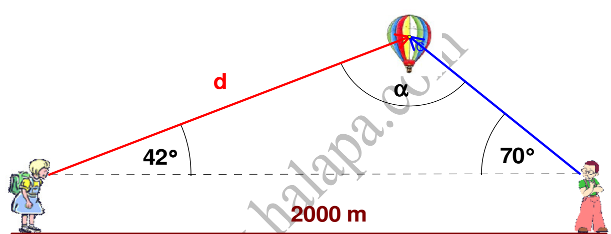
$$\alpha+\beta+\gamma=180^0.$$

Poučak o sinusima (sinusov poučak)

U trokutu ABC vrijedi

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma} \quad \text{ili} \quad a:b:c=\sin\alpha:\sin\beta:\sin\gamma.$$

Gledaj sliku!



Najprije odredimo kut α:

$$\alpha+42^0+70^0=180^0 \Rightarrow \alpha=180^0-42^0-70^0 \Rightarrow \alpha=68^0.$$

Pomoću sinusova poučka izračuna se udaljenost d:

$$\frac{d}{\sin 70^0}=\frac{2000}{\sin 68^0} \Rightarrow \frac{d}{\sin 70^0}=\frac{2000}{\sin 68^0} \;/\cdot\sin 70^0 \Rightarrow d=\frac{2000}{\sin 68^0}\cdot\sin 70^0 \Rightarrow d=2026.98\ m.$$

Vježba 124

Mate i Sanja udaljeni su 2000 m i oboje gledaju balon. U isto vrijeme, Sanja vidi balon pod kutom 42º, a Mate pod kutom 80º. Koliko je Sanja udaljena od balona?

Rezultat: 2124.30 m.

Zadatak 125 (Goran, gimnazija)

Izračunaj $\cos(\alpha-\beta)$ ako je $\sin\alpha+\sin\beta=1$ i $\cos\alpha+\cos\beta=\sqrt{2}$.

Rješenje 125

Ponovimo!

$$(x+y)^2=x^2+2\cdot x\cdot y+y^2 \quad , \quad \cos^2 x+\sin^2 x=1 \quad , \quad \cos(x-y)=\cos x\cdot\cos y+\sin x\cdot\sin y.$$

$$\left.\begin{array}{l}\sin\alpha+\sin\beta=1\\ \cos\alpha+\cos\beta=\sqrt{2}\end{array}\right\} \Rightarrow \left[\begin{array}{l}\text{kvadriramo}\\ \text{jednakosti}\end{array}\right] \Rightarrow \left.\begin{array}{l}\sin\alpha+\sin\beta=1 \;/^2\\ \cos\alpha+\cos\beta=\sqrt{2} \;/^2\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \sin^2\alpha + 2\cdot\sin\alpha\cdot\sin\beta + \sin^2\beta = 1 \\ \cos^2\alpha + 2\cdot\cos\alpha\cdot\cos\beta + \cos^2\beta = 2 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{zbrojimo} \\ \text{jednakosti} \end{array}\right] \Rightarrow$$

$$\Rightarrow \sin^2\alpha + \cos^2\alpha + 2\cdot\sin\alpha\cdot\sin\beta + 2\cdot\cos\alpha\cdot\cos\beta + \sin^2\beta + \cos^2\beta = 3 \Rightarrow$$

$$\Rightarrow \left(\sin^2\alpha + \cos^2\alpha\right) + 2\cdot(\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta) + \left(\sin^2\beta + \cos^2\beta\right) = 3 \Rightarrow 1 + 2\cdot\cos(\alpha-\beta) + 1 = 3 \Rightarrow$$

$$\Rightarrow 2\cdot\cos(\alpha-\beta) = 3 - 2 \Rightarrow 2\cdot\cos(\alpha-\beta) = 1 \;/:2 \Rightarrow \cos(\alpha-\beta) = \frac{1}{2}.$$

Vježba 125

Izračunaj $\cos(\alpha-\beta)$ ako je $\sin\alpha - \sin\beta = 1$ i $\cos\alpha - \cos\beta = \sqrt{2}.$

Rezultat: $-\frac{1}{2}.$

Zadatak 126 (Marina, gimnazija)

Dokaži identitet: $\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta.$

Rješenje 126

Ponovimo!

$$\sin(x+y) = \sin x\cdot\cos y + \cos x\cdot\sin y \quad,\quad \sin(x-y) = \sin x\cdot\cos y - \cos x\cdot\sin y \quad,\quad x^2 - y^2 = (x-y)\cdot(x+y).$$

$$\cos^2 x + \sin^2 x = 1 \quad,\quad a^n\cdot b^n = (a\cdot b)^n \quad,\quad \sin 2x = 2\cdot\sin x\cdot\cos x.$$

$$\sin x + \sin y = 2\cdot\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2} \quad,\quad \sin x - \sin y = 2\cdot\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2}.$$

1.inačica

$$\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta.$$

Transformiramo lijevu stranu identiteta da bismo dobili desnu stranu:

$$\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) = (\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta)\cdot(\sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta) = [\text{razlika kvadrata}] =$$

$$= (\sin\alpha\cdot\cos\beta)^2 - (\cos\alpha\cdot\sin\beta)^2 = \sin^2\alpha\cdot\cos^2\beta - \cos^2\alpha\cdot\sin^2\beta = \sin^2\alpha\cdot\left(1-\sin^2\beta\right) - \left(1-\sin^2\alpha\right)\cdot\sin^2\beta =$$

$$= \sin^2\alpha - \sin^2\alpha\cdot\sin^2\beta - \sin^2\beta + \sin^2\alpha\cdot\sin^2\beta = \sin^2\alpha - \sin^2\beta.$$

2.inačica

$$\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta.$$

Transformiramo desnu stranu identiteta da bismo dobili lijevu stranu:

$$\sin^2\alpha - \sin^2\beta = (\sin\alpha - \sin\beta)\cdot(\sin\alpha + \sin\beta) =$$

$$= 2\cdot\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\cdot 2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} = \left(2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\right)\cdot\left(2\cdot\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\right) =$$

$$= \sin 2\cdot\frac{\alpha+\beta}{2}\cdot\sin 2\cdot\frac{\alpha-\beta}{2} = \sin(\alpha+\beta)\cdot\sin(\alpha-\beta).$$

Vježba 126

Dokaži identitet: $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta.$

Rezultat: Dokaz analogan.

Zadatak 127 (Vlado, srednja škola)

Riješi jednadžbu: $\sin^6 x+\cos^6 x=\frac{3}{4}.$

Rješenje 127

Ponovimo!

$$\left(a^n\right)^m=a^{n\cdot m} \quad , \quad a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right) \quad , \quad a^2+2\cdot a\cdot b+b^2=(a+b)^2.$$

$$\cos^2\alpha+\sin^2\alpha=1 \quad , \quad \sin^2\alpha=\frac{1-\cos 2\alpha}{2}.$$

Lijevu stranu jednadžbe transformiramo uporabom formule za zbroj kubova i kvadrat zbroja:

$$\sin^6 x+\cos^6 x=\frac{3}{4} \Rightarrow \left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3=\frac{3}{4} \Rightarrow$$

$$\Rightarrow \left(\sin^2 x+\cos^2 x\right)\cdot\left(\left(\sin^2 x\right)^2-\sin^2 x\cdot\cos^2 x+\left(\cos^2 x\right)^2\right)=\frac{3}{4} \Rightarrow$$

$$\Rightarrow 1\cdot\left(\sin^4 x-\sin^2 x\cdot\cos^2 x+\cos^4 x\right)=\frac{3}{4} \Rightarrow \sin^4 x-\sin^2 x\cdot\cos^2 x+\cos^4 x=\frac{3}{4} \Rightarrow$$

$$\Rightarrow \sin^4 x+2\cdot\sin^2 x\cdot\cos^2 x+\cos^4 x-3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4} \Rightarrow$$

$$\Rightarrow \left(\sin^4 x+2\cdot\sin^2 x\cdot\cos^2 x+\cos^4 x\right)-3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4} \Rightarrow \left(\sin^2 x+\cos^2 x\right)^2-3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4} \Rightarrow$$

$$\Rightarrow 1^2-3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4} \Rightarrow 1-3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4} \Rightarrow -3\cdot\sin^2 x\cdot\cos^2 x=\frac{3}{4}-1 \Rightarrow$$

$$\Rightarrow -3\cdot\sin^2 x\cdot\cos^2 x=-\frac{1}{4} \Rightarrow -3\cdot\frac{1}{4}\cdot 4\cdot\sin^2 x\cdot\cos^2 x=-\frac{1}{4} \Rightarrow -3\cdot\frac{1}{4}\cdot(2\cdot\sin x\cdot\cos x)^2=-\frac{1}{4} \Rightarrow$$

$$\Rightarrow -\frac{3}{4}\cdot(2\cdot\sin x\cdot\cos x)^2=-\frac{1}{4} \;/\cdot\left(-\frac{4}{3}\right) \Rightarrow (2\cdot\sin x\cdot\cos x)^2=\frac{1}{3} \Rightarrow \sin^2 2x=\frac{1}{3} \Rightarrow \frac{1-\cos 4x}{2}=\frac{1}{3} \Rightarrow$$

$$\Rightarrow \frac{1-\cos 4x}{2}=\frac{1}{3} \;/\cdot 2 \Rightarrow 1-\cos 4x=\frac{2}{3} \Rightarrow -\cos 4x=\frac{2}{3}-1 \Rightarrow -\cos 4x=-\frac{1}{3} \;/\cdot(-1) \Rightarrow$$

$$\Rightarrow \cos 4x=\frac{1}{3} \Rightarrow 4\cdot x=\cos^{-1}\left(\frac{1}{3}\right) \Rightarrow \left[\begin{matrix}\text{računamo u}\\ \text{radijanima}\end{matrix}\right] \Rightarrow 4\cdot x=\pm 1.230959417+k\cdot 2\cdot\pi \;/:4 \Rightarrow$$

$$\Rightarrow x_{1,2}=\pm 0.307739854+k\cdot\frac{\pi}{2} \;,\; k\in Z.$$

Vježba 127

Riješi jednadžbu: $\sin^4 x-\cos^4 x=1.$

Rezultat: $x=\frac{\pi}{2}+k\cdot\pi=(2\cdot k+1)\cdot\frac{\pi}{2} \;,\; k\in Z.$

Zadatak 128 (3A, TUPŠ)

Izračunaj bez uporabe računala: cos 80º · cos 20º + cos 10º · cos 70º.

Rješenje 128

Ponovimo!

$$\cos\left(90^0-x\right)=\sin x \quad , \quad \text{Adicijska formula: } \cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y.$$

Uporabom adicijske formule dobije se:

cos 80º · cos 20º + cos 10º · cos 70º = cos 80º · cos 20º + cos (90º – 80º) · cos (90º – 20º) =

= cos 80º · cos 20º + sin 80º · sin 20º = cos (80º – 20º) = cos 60º = 0.5.

Vježba 128

Izračunaj bez uporabe računala: cos 70º · cos 10º + cos 20º · cos 80º.

Rezultat: 0.5.

Zadatak 129 (Boby, tehnička škola)

Pojednostavni: $\frac{\sin 2x}{1+\cos 2x}$.

Rješenje 129

Ponovimo!

$$\sin 2\alpha = 2\cdot\sin\alpha\cdot\cos\alpha \quad , \quad \cos^2\alpha+\sin^2\alpha=1 \quad , \quad \cos 2\alpha=\cos^2\alpha-\sin^2\alpha.$$

$$\frac{\sin 2x}{1+\cos 2x}=\frac{2\cdot\sin\alpha\cdot\cos\alpha}{\cos^2\alpha+\sin^2\alpha+\cos^2\alpha-\sin^2\alpha}=\frac{2\cdot\sin\alpha\cdot\cos\alpha}{\cos^2\alpha+\sin^2\alpha+\cos^2\alpha-\sin^2\alpha}=\frac{2\cdot\sin\alpha\cdot\cos\alpha}{2\cdot\cos^2\alpha}=$$

$$=\frac{2\cdot\sin\alpha\cdot\cos\alpha}{2\cdot\cos^2\alpha}=\frac{\sin\alpha}{\cos\alpha}=tg\,\alpha.$$

Vježba 129

Pojednostavni: $\frac{1+\cos 2x}{\sin 2x}$.

Rezultat: ctg x.

Zadatak 130 (Ljubica, gimnazija)

Dokaži da za svaki šiljasti kut α vrijedi relacija: sin α + cos α > 1.

Rješenje 130

Ponovimo!

$$|x|=x \text{ za } x>0 \quad , \quad \sqrt{a^2}=|a| \quad , \quad (a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad \cos^2\alpha+\sin^2\alpha=1.$$

$$\sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha \quad , \quad 0<\alpha<90^0 \Rightarrow 0<\sin\alpha<1.$$

$$\sin\alpha+\cos\alpha=|\sin\alpha+\cos\alpha|=\sqrt{(\sin\alpha+\cos\alpha)^2}=\sqrt{\sin^2\alpha+2\cdot\sin\alpha\cdot\cos\alpha+\cos^2\alpha}=$$

$$=\sqrt{1+2\cdot\sin\alpha\cdot\cos\alpha}=\sqrt{1+\sin 2\alpha}=\begin{bmatrix}0<\alpha<90^0\\ \sin 2\alpha>0\end{bmatrix}=\sqrt{1+\sin 2\alpha}>1.$$

Vježba 130

Dokaži da za svaki šiljasti kut α vrijedi relacija: $\cos^2$ α + $\sin^2$ α – cos α – sin α < 0.

Rezultat: Dokaz analogan.

Zadatak 131 (Vesna, gimnazija)

Koja relacija postoji među šiljastim kutovima α i β ako vrijedi: $\frac{1-tg\,\beta}{1+tg\,\beta}=tg\,\alpha$?

Rješenje 131

Ponovimo!

$$tg\frac{\pi}{4}=1 \quad , \quad tg(\alpha-\beta)=\frac{tg\,\alpha-tg\,\beta}{1+tg\,\alpha\cdot tg\,\beta}.$$

$$tg\,\alpha=\frac{1-tg\,\beta}{1+tg\,\beta}=\frac{tg\frac{\pi}{4}-tg\,\beta}{1+tg\,\beta}=\frac{tg\frac{\pi}{4}-tg\,\beta}{1+1\cdot tg\,\beta}=\frac{tg\frac{\pi}{4}-tg\,\beta}{1+tg\frac{\pi}{4}\cdot tg\,\beta}=tg\left(\frac{\pi}{4}-\beta\right).$$

Iz

$$tg\,\alpha = tg\left(\frac{\pi}{4} - \beta\right)$$

dobije se:

$$\alpha = \frac{\pi}{4} - \beta \;\Rightarrow\; \alpha + \beta = \frac{\pi}{4}.$$

Vježba 131

Koja relacija postoji među šiljastim kutovima α i β ako vrijedi: $\frac{1 + tg\,\beta}{1 - tg\,\beta} = \frac{1}{tg\,\alpha}$?

Rezultat: $\alpha + \beta = \frac{\pi}{4}.$

Zadatak 132 (Josip, srednja škola)

Dokaži jednakost: $\frac{1}{\sin 10^0} - 4 \cdot \sin 70^0 = 2.$

Rješenje 132

Ponovimo!

$$\sin x \cdot \sin y = \frac{1}{2} \cdot \left[\cos(x - y) - \cos(x + y)\right] \quad , \quad \cos\left(90^0 - \alpha\right) = \sin\alpha.$$

$$\frac{1}{\sin 10^0} - 4 \cdot \sin 70^0 = 2 \;\Rightarrow\; \frac{1 - 4 \cdot \sin 70^0 \cdot \sin 10^0}{\sin 10^0} = 2 \;\Rightarrow$$

$$\Rightarrow \frac{1 - 4 \cdot \frac{1}{2} \cdot \left[\cos\left(70^0 - 10^0\right) - \cos\left(70^0 + 10^0\right)\right]}{\sin 10^0} = 2 \;\Rightarrow\; \frac{1 - 2 \cdot \left[\cos 60^0 - \cos 80^0\right]}{\sin 10^0} = 2 \;\Rightarrow$$

$$\Rightarrow \frac{1 - 2 \cdot \left[\frac{1}{2} - \cos 80^0\right]}{\sin 10^0} = 2 \;\Rightarrow\; \frac{1 - 1 + 2 \cdot \cos 80^0}{\sin 10^0} = 2 \;\Rightarrow\; \frac{1 - 1 + 2 \cdot \cos 80^0}{\sin 10^0} = 2 \;\Rightarrow\; \frac{2 \cdot \cos 80^0}{\sin 10^0} = 2 \;\Rightarrow$$

$$\Rightarrow \frac{2 \cdot \sin 10^0}{\sin 10^0} = 2 \;\Rightarrow\; \frac{2 \cdot \sin 10^0}{\sin 10^0} = 2 \;\Rightarrow\; 2 = 2.$$

Vježba 132

Dokaži jednakost: $\frac{1}{\cos 80^0} - 4 \cdot \cos 20^0 = 2.$

Rezultat: Dokaz analogan.

Zadatak 133 (Vedran, gimnazija)

Za koji realan broj m jednadžba $\sqrt{3} \cdot \sin x + \cos x = m$ ima realna rješenja?

Rješenje 133

Ponovimo!

Za svaki realni broj x vrijedi:

$$|\cos x| \le 1 \text{ ili } \cos x \in [-1, 1] \text{ ili } -1 \le \cos x \le 1.$$

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y.$$

Transformiramo zadanu jednadžbu:

$$\sqrt{3}\cdot\sin x+\cos x=m \Rightarrow tg\frac{\pi}{3}\cdot\sin x+\cos x=m \Rightarrow \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}\cdot\sin x+\cos x=m \;/\cdot\cos\frac{\pi}{3} \Rightarrow$$

$$\Rightarrow \sin\frac{\pi}{3}\cdot\sin x+\cos\frac{\pi}{3}\cdot\cos x=m\cdot\cos\frac{\pi}{3} \Rightarrow \cos x\cdot\cos\frac{\pi}{3}+\sin x\cdot\sin\frac{\pi}{3}=\frac{1}{2}\cdot m \Rightarrow \cos\left(x-\frac{\pi}{3}\right)=\frac{1}{2}\cdot m.$$

Budući da je

$$-1\le\cos x\le 1,$$

slijedi:

$$-1\le\frac{1}{2}\cdot m\le 1 \Rightarrow -1\le\frac{1}{2}\cdot m\le 1 \;/\cdot 2 \Rightarrow -2\le m\le 2 \Rightarrow m\in[-2,\,2].$$

Vježba 133

Za koji realan broj m jednadžba $\sqrt{3}\cdot\sin x+\cos x=\frac{1}{2}\cdot m$ ima realna rješenja?

Rezultat: $m\in[-4,\,4]$.

Zadatak 134 (Josip, gimnazija)

Izračunaj 3 · tg x + 3 · ctg x, ako je 2 · sin x + 2 · cos x = 1.

Rješenje 134

Ponovimo!

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2 \quad , \quad \cos^2\alpha+\sin^2\alpha=1.$$

$$2\cdot\sin x+2\cdot\cos x=1 \Rightarrow 2\cdot\sin x+2\cdot\cos x=1 \;/:2 \Rightarrow \sin x+\cos x=\frac{1}{2} \;/^2 \Rightarrow (\sin x+\cos x)^2=\left(\frac{1}{2}\right)^2 \Rightarrow$$

$$\Rightarrow \sin^2 x+2\cdot\sin x\cdot\cos x+\cos^2 x=\frac{1}{4} \Rightarrow 1+2\cdot\sin x\cdot\cos x=\frac{1}{4} \Rightarrow 2\cdot\sin x\cdot\cos x=\frac{1}{4}-1 \Rightarrow$$

$$\Rightarrow 2\cdot\sin x\cdot\cos x=-\frac{3}{4} \;/:2 \Rightarrow \sin x\cdot\cos x=-\frac{3}{8}.$$

Računamo zadani izraz:

$$3\cdot tg\,x+3\cdot ctg\,x=3\cdot\frac{\sin x}{\cos x}+3\cdot\frac{\cos x}{\sin x}=3\cdot\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)=3\cdot\frac{\sin^2 x+\cos^2 x}{\sin x\cdot\cos x}=3\cdot\frac{1}{\sin x\cdot\cos x}=$$

$$=\left[\sin x\cdot\cos x=-\frac{3}{8}\right]=3\cdot\frac{1}{-\frac{3}{8}}=3\cdot\left(-\frac{8}{3}\right)=-8.$$

Vježba 134

Izračunaj 6 · tg x + 6 · ctg x, ako je 2 · sin x + 2 · cos x = 1.

Rezultat: – 16.

Zadatak 135 (Natalija, gimnazija)

Pokaži da je trokut jednakokračan ako za njegove kutove vrijedi relacija: $2\cdot\cos\alpha=\frac{\sin\gamma}{\sin\beta}$.

Rješenje 135

Ponovimo!

Zbroj svih kutova u trokutu je 180º:

$$\alpha+\beta+\gamma=180^0.$$

Kod jednakokračnog trokuta duljine dviju stranica su jednake. Stranice jednake duljine zovemo kraci trokuta. Uočimo da su kutovi koji leže na trećoj stranici jednaki zbog činjenice da se nasuprot jednakim stranicama nalaze jednaki kutovi.

$$\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y \quad , \quad \sin(x-y)=\sin x\cdot\cos y-\cos x\cdot\sin y \quad , \quad \sin\left(180^0-x\right)=\sin x.$$

$$2\cdot\cos\alpha=\frac{\sin\gamma}{\sin\beta} \Rightarrow 2\cdot\cos\alpha=\frac{\sin\gamma}{\sin\beta} \;/\cdot\sin\beta \Rightarrow 2\cdot\cos\alpha\cdot\sin\beta=\sin\gamma \Rightarrow \begin{bmatrix}\alpha+\beta+\gamma=180^0\\ \gamma=180^0-(\alpha+\beta)\end{bmatrix} \Rightarrow$$

$$\Rightarrow 2\cdot\cos\alpha\cdot\sin\beta=\sin\left(180^0-(\alpha+\beta)\right) \Rightarrow 2\cdot\cos\alpha\cdot\sin\beta=\sin(\alpha+\beta) \Rightarrow$$

$$\Rightarrow 2\cdot\cos\alpha\cdot\sin\beta=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta \Rightarrow \sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta-2\cdot\cos\alpha\cdot\sin\beta=0 \Rightarrow$$

$$\Rightarrow \sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta=0 \Rightarrow \sin(\alpha-\beta)=0 \Rightarrow \alpha-\beta=0 \Rightarrow \alpha=\beta.$$

Vježba 135

Pokaži da je trokut jednakokračan ako za njegove kutove vrijedi relacija: $2\cdot\cos\beta=\frac{\sin\alpha}{\sin\gamma}$.

Rezultat: β = γ.

Zadatak 136 (Emy, gimnazija)

Ako kutovi trokuta ABC zadovoljavaju jednakost $\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$, dokaži da je trokut ABC pravokutan ili jednakokračan.

Rješenje 136

Ponovimo!

$$x^2-y^2=(x-y)\cdot(x+y).$$

Zbroj svih kutova u trokutu je 180º:

$$\alpha+\beta+\gamma=180^0.$$

Pravokutan trokut ima jedan pravi kut. Pravi kut iznosi 90º.
Kod jednakokračnog trokuta duljine dviju stranica su jednake. Stranice jednake duljine zovemo kraci trokuta. Uočimo da su kutovi koji leže na trećoj stranici jednaki zbog činjenice da se nasuprot jednakim stranicama nalaze jednaki kutovi.

$$\sin x+\sin y=2\cdot\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2} \quad , \quad \sin x-\sin y=2\cdot\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2} \quad , \quad \sin 2x=2\cdot\sin x\cdot\cos x.$$

$$x\cdot y=0 \Rightarrow x=0 \text{ ili } y=0 \text{ ili } x=y=0.$$

$$\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta \Rightarrow \sin(\alpha-\beta)=(\sin\alpha-\sin\beta)\cdot(\sin\alpha+\sin\beta) \Rightarrow$$

$$\Rightarrow \sin(\alpha-\beta)=2\cdot\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\cdot 2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \Rightarrow$$

$$\Rightarrow \sin(\alpha-\beta)=\left(2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\right)\cdot\left(2\cdot\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\right) \Rightarrow$$

$$\Rightarrow \sin(\alpha-\beta)=\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) \Rightarrow \sin(\alpha-\beta)-\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)=0 \Rightarrow$$

$$\Rightarrow \sin(\alpha-\beta)\cdot\left[1-\sin(\alpha+\beta)\right]=0 \Rightarrow \left.\begin{matrix}\sin(\alpha-\beta)=0\\ 1-\sin(\alpha+\beta)=0\end{matrix}\right\} \Rightarrow \left.\begin{matrix}\sin(\alpha-\beta)=0\\ \sin(\alpha+\beta)=1\end{matrix}\right\} \Rightarrow \left.\begin{matrix}\alpha-\beta=0\\ \alpha+\beta=90^0\end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix}\alpha=\beta \; - \text{ trokut } ABC \text{ je jednakokračan}\\ \gamma=90^0 \; - \text{ trokut } ABC \text{ je pravokutan}\end{matrix}\right\}.$$

Vježba 136

Ako kutovi trokuta ABC zadovoljavaju jednakost $\sin(\alpha-\gamma)=\sin^2\alpha-\sin^2\gamma$, dokaži da je trokut ABC pravokutan ili jednakokračan.

Rezultat: Dokaz analogan.

Zadatak 137 (3A, TUPŠ)

Riješi sustav jednadžbi: $\begin{cases} x-y=\dfrac{\pi}{6} \\ \cos x\cdot\cos y=\dfrac{\sqrt{3}}{4}. \end{cases}$

Rješenje 137

Ponovimo!

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}\cdot\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right].$$

Transformiramo trigonometrijsku jednadžbu u zbroj.

$$\left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \cos x\cdot\cos y=\dfrac{\sqrt{3}}{4} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \dfrac{1}{2}\cdot\left[\cos(x-y)+\cos(x+y)\right]=\dfrac{\sqrt{3}}{4} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \dfrac{1}{2}\cdot\left[\cos(x-y)+\cos(x+y)\right]=\dfrac{\sqrt{3}}{4} \;/\cdot 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \cos(x-y)+\cos(x+y)=\dfrac{\sqrt{3}}{2} \end{array}\right\}.$$

Umjesto x – y u drugu jednadžbu uvrstimo $\dfrac{\pi}{6}$.

$$\Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \cos\dfrac{\pi}{6}+\cos(x+y)=\dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \dfrac{\sqrt{3}}{2}+\cos(x+y)=\dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \dfrac{\sqrt{3}}{2}+\cos(x+y)=\dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ \cos(x+y)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ x+y=\dfrac{\pi}{2}+k\cdot\pi \end{array}\right\}.$$

Zajedno s prvom jednadžbom dobije se sustav linearnih jednadžbi s dvije nepoznanice.

$$\left.\begin{array}{l} x-y=\dfrac{\pi}{6} \\ x+y=\dfrac{\pi}{2}+k\cdot\pi \end{array}\right\} \Rightarrow \left[\begin{array}{l} \text{metoda} \\ \text{suprotnih} \\ \text{koeficijenata} \end{array}\right] \Rightarrow 2\cdot x=\frac{\pi}{6}+\frac{\pi}{2}+k\cdot\pi \Rightarrow 2\cdot x=\frac{\pi+3\cdot\pi}{6}+k\cdot\pi \Rightarrow$$

$$\Rightarrow 2\cdot x=\frac{4\cdot\pi}{6}+k\cdot\pi \Rightarrow 2\cdot x=\frac{2\cdot\pi}{3}+k\cdot\pi \;/\cdot\frac{1}{2} \Rightarrow x=\frac{\pi}{3}+k\cdot\frac{\pi}{2}.$$

Nepoznanicu y izračunamo, na primjer, iz jednadžbe:

$$x+y=\frac{\pi}{2}+k\cdot\pi.$$

$$\left.\begin{array}{l} x=\dfrac{\pi}{3}+k\cdot\dfrac{\pi}{2} \\ x+y=\dfrac{\pi}{2}+k\cdot\pi \end{array}\right\} \Rightarrow \frac{\pi}{3}+k\cdot\frac{\pi}{2}+y=\frac{\pi}{2}+k\cdot\pi \Rightarrow y=\frac{\pi}{2}-\frac{\pi}{3}+k\cdot\pi-k\cdot\frac{\pi}{2} \Rightarrow$$

$$\Rightarrow\ y=\frac{3\cdot\pi-2\cdot\pi}{6}+k\cdot\left(\pi-\frac{\pi}{2}\right)\ \Rightarrow\ y=\frac{\pi}{6}+k\cdot\frac{\pi}{2}.$$

Rješenje sustava jednadžbi je skup uređenih parova:

$$\left\{(x,\ y)\ \mid\ x=\frac{\pi}{3}+k\cdot\frac{\pi}{2},\ y=\frac{\pi}{6}+k\cdot\frac{\pi}{2},\ k\in Z\right\}.$$

Vježba 137

Riješi sustav jednadžbi: $\begin{cases} x-y=\dfrac{2\cdot\pi}{3} \\ \sin x+\sin y=1. \end{cases}$

Rezultat: $\left\{(x,\ y)\ \mid\ x=\frac{5\cdot\pi}{6}+k\cdot 2\cdot\pi,\ y=\frac{\pi}{6}+k\cdot 2\cdot\pi,\ k\in Z\right\}.$

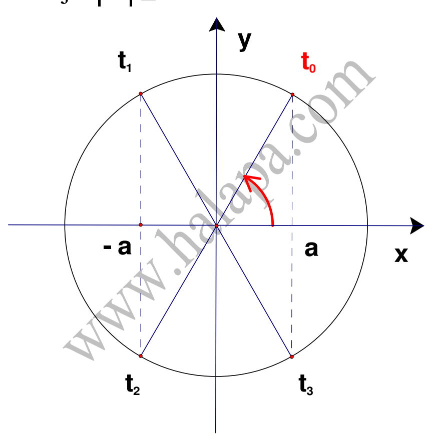
Zadatak 138 (Ante, Visoka škola za sigurnost)

Riješi jednadžbu: $-2\cdot\cos(x-\pi)-1=0\cdot$

Rješenje 138

Ponovimo!

Jednadžba cos x = a ima rješenja ako je $|\,a\,|\le 1$.



Pomoću $|\,a\,|$ koristeći tablice ili kalkulator odredimo t_0:

$$t_0=\cos^{-1}|\,a\,|.$$

Ako je točka u prvom kvadrantu onda je glavna mjera kuta t_0 pa je rješenje

$$x_1=t_0+k\cdot 2\cdot\pi,\ k\in Z.$$

Ako je točka u drugom kvadrantu onda je glavna mjera kuta $t_1=\pi-t_0$ pa je rješenje

$$x_2=\pi-t_0+k\cdot 2\cdot\pi,\ k\in Z.$$

Ako je točka u trećem kvadrantu onda je glavna mjera kuta $t_2=\pi+t_0$ pa je rješenje

$$x_3=\pi+t_0+k\cdot 2\cdot\pi,\ k\in Z.$$

Ako je točka u četvrtom kvadrantu onda je glavna mjera kuta $t_3=2\cdot\pi-t_0$ pa je rješenje

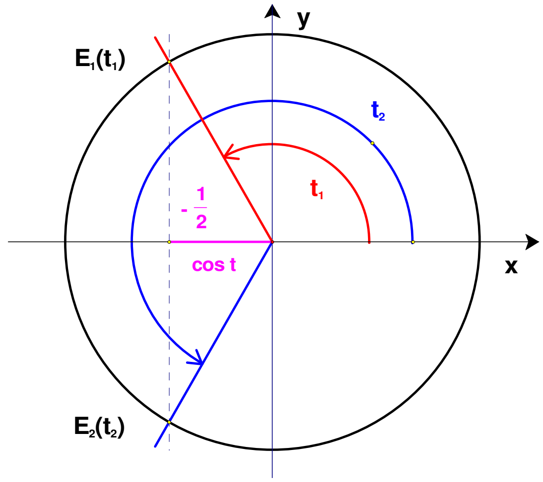
$$x_4=2\cdot\pi-t_0+k\cdot 2\cdot\pi,\ k\in Z.$$

Tako dobijemo rješenja jednadžbe cos x = a:

$$x_1=t_0+k\cdot 2\cdot\pi\quad,\quad x_2=2\cdot\pi-t_0+k\cdot 2\cdot\pi,\ k\in Z,\ \text{za}\ a>0,$$

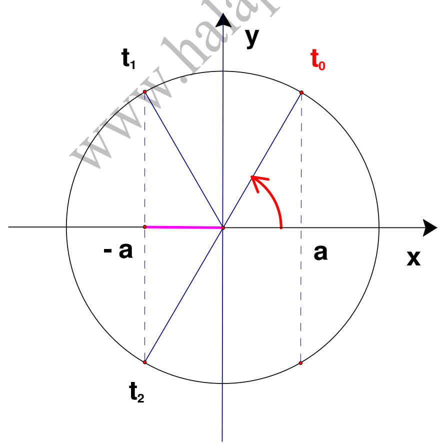
$$x_1=\pi-t_0+k\cdot 2\cdot\pi\quad,\quad x_2=\pi+t_0+k\cdot 2\cdot\pi,\ k\in Z,\ \text{za}\ a<0.$$

$$-2\cdot\cos(x-\pi)-1=0 \Rightarrow -2\cdot\cos(x-\pi)=1\ /:(-2) \Rightarrow \cos(x-\pi)=-\frac{1}{2} \Rightarrow \begin{bmatrix}\text{supstitucija}\\ t=x-\pi\end{bmatrix} \Rightarrow \cos t=-\frac{1}{2}.$$



Na slici točkom $\left(-\frac{1}{2},\ 0\right)$ položimo pravac $x=-\frac{1}{2}$ koji trigonometrijsku kružnicu siječe u dvije točke $E_1(t_1)$ i $E_2(t_2)$. Simetrijom je svakoj točki koja nije u prvom kvadrantu pridružena točka u prvom kvadrantu kojoj je pridružen broj

$$t_0\in\left\langle 0,\ \frac{\pi}{2}\right\rangle.$$



$$\left|-\frac{1}{2}\right|=\frac{1}{2} \Rightarrow t_0=\cos^{-1}\left(\frac{1}{2}\right) \Rightarrow t_0=\frac{\pi}{3}.$$

Zato je:

- $t_1=\pi-t_0+k\cdot2\cdot\pi,\ k\in Z \Rightarrow t_1=\pi-\frac{\pi}{3}+k\cdot2\cdot\pi,\ k\in Z \Rightarrow t_1=\frac{2\cdot\pi}{3}+k\cdot2\cdot\pi,\ k\in Z.$
- $t_2=\pi+t_0+k\cdot2\cdot\pi,\ k\in Z \Rightarrow t_2=\pi+\frac{\pi}{3}+k\cdot2\cdot\pi,\ k\in Z \Rightarrow t_2=\frac{4\cdot\pi}{3}+k\cdot2\cdot\pi,\ k\in Z.$

Vraćamo se na supstituciju:

$$t=x-\pi \Rightarrow x=\pi+t.$$

Rješenja iznose:

$$\left.\begin{array}{l} t_1 = \frac{2 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z \\ x_1 = \pi + t_1 \end{array}\right\} \Rightarrow x_1 = \pi + \frac{2 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z \Rightarrow x_1 = \frac{5 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z,$$

$$\left.\begin{array}{l} t_2 = \frac{4 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z \\ x_2 = \pi + t_2 \end{array}\right\} \Rightarrow x_2 = \pi + \frac{4 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z \Rightarrow x_2 = \frac{7 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z.$$

Vježba 138

Riješi jednadžbu: $2 \cdot \cos(x - \pi) + 1 = 0 \cdot$

Rezultat: $x_1 = \frac{5 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z\ ,\ x_2 = \frac{7 \cdot \pi}{3} + k \cdot 2 \cdot \pi,\ k \in Z.$

Zadatak 139 (Tina, gimnazija)

Ako kutovi α, β i γ nekog trokuta zadovoljavaju jednakost: $\sin\alpha = \frac{\sin\beta + \sin\gamma}{\cos\beta + \cos\gamma}$, nađi kut α.

Rješenje 139

Ponovimo!

Zbroj kutova u trokutu je 180°:

$$\alpha + \beta + \gamma = 180^0.$$

$$\frac{a-b}{n} = \frac{a}{n} - \frac{b}{n} \quad , \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad , \quad \sin\left(90^0 - x\right) = \cos x \quad , \quad \cos\left(90^0 - x\right) = \sin x.$$

$$\sin x = 2 \cdot \sin\frac{x}{2} \cdot \cos\frac{x}{2} \quad , \quad x \cdot y = 0 \Rightarrow x = 0 \text{ ili } y = 0 \text{ ili } x = y = 0.$$

Formule za transformaciju:

$$\sin x + \sin y = 2 \cdot \sin\frac{x+y}{2} \cdot \cos\frac{x-y}{2} \quad , \quad \cos x + \cos y = 2 \cdot \cos\frac{x+y}{2} \cdot \cos\frac{x-y}{2}.$$

$$\sin\alpha = \frac{\sin\beta + \sin\gamma}{\cos\beta + \cos\gamma} \Rightarrow \sin\alpha = \frac{2 \cdot \sin\frac{\beta+\gamma}{2} \cdot \cos\frac{\beta-\gamma}{2}}{2 \cdot \cos\frac{\beta+\gamma}{2} \cdot \cos\frac{\beta-\gamma}{2}} \Rightarrow \sin\alpha = \frac{2 \cdot \sin\frac{\beta+\gamma}{2} \cdot \cos\frac{\beta-\gamma}{2}}{2 \cdot \cos\frac{\beta+\gamma}{2} \cdot \cos\frac{\beta-\gamma}{2}} \Rightarrow$$

$$\Rightarrow \sin\alpha = \frac{\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta+\gamma}{2}} \Rightarrow \sin\alpha = \frac{\sin\frac{180^0 - \alpha}{2}}{\cos\frac{180^0 - \alpha}{2}} \Rightarrow \sin\alpha = \frac{\sin\left(\frac{180^0}{2} - \frac{\alpha}{2}\right)}{\cos\left(\frac{180^0}{2} - \frac{\alpha}{2}\right)} \Rightarrow$$

$$\Rightarrow \sin\alpha = \frac{\sin\left(90^0 - \frac{\alpha}{2}\right)}{\cos\left(90^0 - \frac{\alpha}{2}\right)} \Rightarrow \sin\alpha = \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} \Rightarrow \sin\alpha = \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} \ / \cdot \sin\frac{\alpha}{2} \Rightarrow \sin\alpha \cdot \sin\frac{\alpha}{2} = \cos\frac{\alpha}{2} \Rightarrow$$

$$\Rightarrow 2 \cdot \sin\frac{\alpha}{2} \cdot \cos\frac{\alpha}{2} \cdot \sin\frac{\alpha}{2} = \cos\frac{\alpha}{2} \Rightarrow 2 \cdot \sin^2\frac{\alpha}{2} \cdot \cos\frac{\alpha}{2} - \cos\frac{\alpha}{2} = 0 \Rightarrow \cos\frac{\alpha}{2} \cdot \left(2 \cdot \sin^2\frac{\alpha}{2} - 1\right) = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} \cos\frac{\alpha}{2}=0 \\ 2\cdot\sin^2\frac{\alpha}{2}-1=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \frac{\alpha}{2}=90^0 \\ 2\cdot\sin^2\frac{\alpha}{2}=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \frac{\alpha}{2}=90^0 \;\; /\cdot 2 \\ 2\cdot\sin^2\frac{\alpha}{2}=1 \;\; /:2 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \alpha=180^0 \;\; \text{nema smisla} \\ \sin^2\frac{\alpha}{2}=\frac{1}{2} \;\; /\sqrt{\;} \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \sin\frac{\alpha}{2}=\sqrt{\frac{1}{2}} \Rightarrow \sin\frac{\alpha}{2}=\frac{\sqrt{1}}{\sqrt{2}} \Rightarrow \sin\frac{\alpha}{2}=\frac{1}{\sqrt{2}} \Rightarrow \begin{bmatrix} \text{racionalizacija} \\ \text{nazivnika} \end{bmatrix} \Rightarrow \sin\frac{\alpha}{2}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \Rightarrow$$

$$\Rightarrow \sin\frac{\alpha}{2}=\frac{\sqrt{2}}{2} \Rightarrow \frac{\alpha}{2}=\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) \Rightarrow \frac{\alpha}{2}=45^0 \;\; /\cdot 2 \Rightarrow \alpha=90^0.$$

Vježba 139

Ako kutovi α, β i γ nekog trokuta zadovoljavaju jednakost: $\sin\beta=\frac{\sin\alpha+\sin\gamma}{\cos\alpha+\cos\gamma}$, nađi kut β.

Rezultat: 90°.

Zadatak 140 (Ico, gimnazija)

Koliki je ukupan broj svih rješenja jednadžbe $\sin 2\cdot x=\sin\left(x+\frac{\pi}{6}\right)$ na intervalu $\left\langle -\pi, \pi\right\rangle$?

Rješenje 140

Ponovimo!

Jednadžbu koja sadrži trigonometrijske funkcije nepoznatog broja ili kuta zovemo trigonometrijska jednadžba s jednom nepoznanicom.

$$\sin x=\sin\alpha \Rightarrow \left.\begin{matrix} x_1=\alpha+k\cdot 2\cdot\pi \\ x_2=\pi-\alpha+k\cdot 2\cdot\pi,\; k\in Z \end{matrix}\right\}.$$

Pitamo se za kakve brojeve x i α vrijedi

sin x = sin α.

Zamislimo da je α zadan i da tražimo sve vrijednosti od x za koje vrijedi ta jednakost. Budući da je funkcija sinus periodična s osnovnim periodom 2 · π, jasno je da će ona vrijediti ako je

$$x=\alpha+k\cdot 2\cdot\pi,\; k\in Z.$$

Međutim, kako je

$$\sin(\pi-\alpha)=\sin\alpha,$$

može biti i

$$x=\pi-\alpha+k\cdot 2\cdot\pi,\; k\in Z.$$

Rješavamo zadatak.

- $$\sin 2\cdot x=\sin\left(x+\frac{\pi}{6}\right) \Rightarrow 2\cdot x=x+\frac{\pi}{6}+k\cdot 2\cdot\pi \Rightarrow 2\cdot x-x=\frac{\pi}{6}+k\cdot 2\cdot\pi \Rightarrow x_1=\frac{\pi}{6}+k\cdot 2\cdot\pi.$$

Zbog uvjeta zadatka rješenja x_1 moraju biti između – π i π.

$$-\pi<x_1<\pi \Rightarrow -\pi<\frac{\pi}{6}+k\cdot 2\cdot\pi<\pi \Rightarrow \begin{bmatrix} \text{od sustava nejednadžbi} \\ \text{oduzmemo } \frac{\pi}{6} \end{bmatrix} \Rightarrow -\pi<\frac{\pi}{6}+k\cdot 2\cdot\pi<\pi \;\; /-\frac{\pi}{6} \Rightarrow$$

$$\Rightarrow -\pi-\frac{\pi}{6}<\frac{\pi}{6}+k\cdot 2\cdot\pi-\frac{\pi}{6}<\pi-\frac{\pi}{6} \Rightarrow -\frac{7}{6}\cdot\pi<k\cdot 2\cdot\pi<\frac{5}{6}\cdot\pi \Rightarrow \begin{bmatrix} \text{sustav nejednadžbi} \\ \text{podijelimo sa } 2\cdot\pi \end{bmatrix} \Rightarrow$$

$$\Rightarrow -\frac{7}{6}\cdot\pi<k\cdot 2\cdot\pi<\frac{5}{6}\cdot\pi \;\; /\cdot\frac{1}{2\cdot\pi} \Rightarrow -\frac{7}{6}\cdot\pi\cdot\frac{1}{2\cdot\pi}<k\cdot 2\cdot\pi\cdot\frac{1}{2\cdot\pi}<\frac{5}{6}\cdot\pi\cdot\frac{1}{2\cdot\pi} \Rightarrow$$

$$\Rightarrow -\frac{7}{12} < k < \frac{5}{12} \Rightarrow \left[\begin{matrix} k \text{ mora biti} \\ \text{cijeli broj} \end{matrix}\right] \Rightarrow k = 0. \text{ Postoji 1 rješenje.}$$

- $$\sin 2\cdot x = \sin\left(x + \frac{\pi}{6}\right) \Rightarrow 2\cdot x = \pi - \left(x + \frac{\pi}{6}\right) + k\cdot 2\cdot\pi \Rightarrow 2\cdot x = \pi - x - \frac{\pi}{6} + k\cdot 2\cdot\pi \Rightarrow$$

$$\Rightarrow 2\cdot x + x = \pi - \frac{\pi}{6} + k\cdot 2\cdot\pi \Rightarrow 3\cdot x = \frac{5}{6}\cdot\pi + k\cdot 2\cdot\pi \;/:3 \Rightarrow x_2 = \frac{5}{18}\cdot\pi + k\cdot\frac{2\cdot\pi}{3}.$$

Zbog uvjeta zadatka rješenja x_2 moraju biti između $-\pi$ i π.

$$-\pi < x_2 < \pi \Rightarrow -\pi < \frac{5}{18}\cdot\pi + k\cdot\frac{2\cdot\pi}{3} < \pi \Rightarrow \left[\begin{matrix} \text{od sustava nejednadžbi} \\ \text{oduzmemo } \frac{5}{18}\cdot\pi \end{matrix}\right] \Rightarrow$$

$$\Rightarrow -\pi < \frac{5}{18}\cdot\pi + k\cdot\frac{2\cdot\pi}{3} < \pi \;/-\frac{5}{18}\cdot\pi \Rightarrow -\pi - \frac{5}{18}\cdot\pi < \frac{5}{18}\cdot\pi + k\cdot\frac{2\cdot\pi}{3} - \frac{5}{18}\cdot\pi < \pi - \frac{5}{18}\cdot\pi \Rightarrow$$

$$\Rightarrow -\frac{23}{18}\cdot\pi < k\cdot\frac{2\cdot\pi}{3} < \frac{13}{18}\cdot\pi \Rightarrow \left[\begin{matrix} \text{sustav nejednadžbi} \\ \text{pomnožimo sa } \frac{3}{2\cdot\pi} \end{matrix}\right] \Rightarrow -\frac{23}{18}\cdot\pi < k\cdot\frac{2\cdot\pi}{3} < \frac{13}{18}\cdot\pi \;/\cdot\frac{3}{2\cdot\pi} \Rightarrow$$

$$\Rightarrow -\frac{23}{18}\cdot\pi\cdot\frac{3}{2\cdot\pi} < k\cdot\frac{2\cdot\pi}{3}\cdot\frac{3}{2\cdot\pi} < \frac{13}{18}\cdot\pi\cdot\frac{3}{2\cdot\pi} \Rightarrow -\frac{23}{12} < k < \frac{13}{12} \Rightarrow \left[\begin{matrix} k \text{ mora biti} \\ \text{cijeli broj} \end{matrix}\right] \Rightarrow$$

$$\Rightarrow k = -1, 0, 1. \text{ Postoje 3 rješenja.}$$

Ukupan broj rješenja je:

$$1 + 3 = 4.$$

Vježba 140

Koliki je ukupan broj svih rješenja jednadžbe $\sin 2\cdot x = \sin\left(x + \frac{\pi}{6}\right)$ na intervalu $\langle 0, \pi\rangle$?

Rezultat: 3.