Zadatak 121 (Anamarija, maturantica TUPŠ-a)

Riješite jednadžbu: $50^{\log x}\cdot 8^{\log x}=400.$

Rješenje 121

Ponovimo!

$a^{x}\cdot b^{x}=(a\cdot b)^{x}\quad,\quad a^{1}=a\quad,\quad \log a=c\ \Rightarrow\ 10^{c}=a\quad,\quad a^{f(x)}=a^{g(x)}\ \Rightarrow\ f(x)=g(x).$

$$50^{\log x}\cdot 8^{\log x}=400\ \Rightarrow\ \begin{bmatrix}\text{Diskusija!}\\ x>0\end{bmatrix}\ \Rightarrow\ (50\cdot 8)^{\log x}=400\ \Rightarrow\ 400^{\log x}=400^{1}\ \Rightarrow\ \log x=1\ \Rightarrow\ x=10^{1}\ \Rightarrow\ x=10.$$

Vježba 121

Riješite jednadžbu: $5^{\log x}\cdot 80^{\log x}=400.$

Rezultat: x = 10.

Zadatak 122 (Anamarija, maturantica TUPŠ-a)

Riješite jednadžbu: $\log x-\log\frac{1}{x-1}-\log 2=\log(2\cdot x+3).$

Rješenje 122

Ponovimo!

$a^{-n}=\frac{1}{a^{n}}\quad,\quad \log a^{n}=n\cdot\log a\quad,\quad \log(a\cdot b)=\log a+\log b\quad,\quad \log f(x)=\log g(x)\ \Rightarrow\ f(x)=g(x).$

Diskusija!

$$\left.\begin{array}{l}x>0\\ x-1>0\\ 2\cdot x+3>0\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}x>0\\ x>1\\ x>-\frac{3}{2}\end{array}\right\}\ \Rightarrow\ x>1.$$

$$\log x-\log\frac{1}{x-1}-\log 2=\log(2\cdot x+3)\ \Rightarrow\ \log x-\log(x-1)^{-1}-\log 2=\log(2\cdot x+3)\ \Rightarrow$$

$$\Rightarrow\ \log x+\log(x-1)=\log(2\cdot x+3)+\log 2\ \Rightarrow\ \log x\cdot(x-1)=\log 2\cdot(2\cdot x+3)\ \Rightarrow\ x\cdot(x-1)=2\cdot(2\cdot x+3)\ \Rightarrow$$

$$\Rightarrow\ x^{2}-x=4\cdot x+6\ \Rightarrow\ x^{2}-x-4\cdot x-6=0\ \Rightarrow\ x^{2}-5\cdot x-6=0\ \Rightarrow\ \left.\begin{array}{l}a=1\ ,\ b=-5\ ,\ c=-6\\ x_{1,2}=\frac{-b\pm\sqrt{b^{2}-4\cdot a\cdot c}}{2\cdot a}\end{array}\right\}\ \Rightarrow$$

$$\Rightarrow\ x_{1,2}=\frac{5\pm\sqrt{25-4\cdot 1\cdot(-6)}}{2\cdot 1}\ \Rightarrow\ x_{1,2}=\frac{5\pm\sqrt{25+24}}{2}\ \Rightarrow\ x_{1,2}=\frac{5\pm\sqrt{49}}{2}\ \Rightarrow\ x_{1,2}=\frac{5\pm 7}{2}\ \Rightarrow$$

$$\Rightarrow\ \left.\begin{array}{l}x_{1}=\frac{5+7}{2}\\ x_{2}=\frac{5-7}{2}\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}x_{1}=\frac{12}{2}\\ x_{2}=\frac{-2}{2}\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}x_{1}=6\ \text{rješenje}\\ x_{2}=-1\ \text{nije rješenje zbog diskusije } x>1\end{array}\right\}.$$

Vježba 122

Riješite jednadžbu: $\log x+\log\frac{1}{2}=\log(2\cdot x+3)+\log\frac{1}{x-1}.$

Rezultat: x = 6.

Zadatak 123 (Anamarija, Sanela, maturantice gimnazije)

Za koji m jednadžba $x^{2}-2\cdot x+\log_{5}m=0$ nema realnih rješenja.

Rješenje 123

Ponovimo!

Ako kvadratna jednadžba $a\cdot x^2+b\cdot x+c=0$ ima kompleksno – konjugirana rješenja njezina je diskriminanta negativan broj: $b^2-4\cdot a\cdot c<0$,

$$\log_b f(x) > \log_b g(x) \ , \ b>1 \Rightarrow f(x) > g(x) \quad , \quad \log_b b = 1.$$

Budući da zadana jednadžba nema realnih rješenja, znači da su joj rješenja kompleksno – konjugirani brojevi pa vrijedi:

$$\left.\begin{array}{l} x^2 - 2\cdot x + \log_5 m = 0 \\ a=1 \ , \ b=-2 \ , \ c=\log_5 m \\ b^2 - 4\cdot a\cdot c < 0 \end{array}\right\} \Rightarrow (-2)^2 - 4\cdot 1\cdot \log_5 m < 0 \Rightarrow 4 - 4\cdot\log_5 m < 0 \Rightarrow -4\cdot\log_5 m < -4 \ /:(-4) \Rightarrow$$

$$\Rightarrow \log_5 m > 1 \Rightarrow \log_5 m > \log_5 5 \Rightarrow m > 5 \Rightarrow m \in \langle 5, +\infty \rangle.$$

Vježba 123

Za koji m jednadžba $x^2 - 2\cdot x + \log_5 m = 0$ ima dvostruko realno rješenje?

Rezultat: m = 5.

Zadatak 124 (Anamarija, maturantica TUPŠ-a)

Riješite jednadžbu: $8^{\log x} + 3^{1-\log x}\cdot 24^{\log(10\cdot x)} = 73.$

Rješenje 124

Ponovimo!

$$\log(a\cdot b) = \log a + \log b \quad , \quad \log_b b = 1 \quad , \quad a^n\cdot a^m = a^{n\cdot m} \quad , \quad a^n\cdot b^n = (a\cdot b)^n$$

$$\log_b 1 = 0 \quad , \quad a^{f(x)} = a^{g(x)} \Rightarrow f(x) = g(x) \quad , \quad a^0 = 1.$$

$$8^{\log x} + 3^{1-\log x}\cdot 24^{\log(10\cdot x)} = 73 \Rightarrow 8^{\log x} + 3^{1-\log x}\cdot(3\cdot 8)^{\log 10 + \log x} = 73 \Rightarrow$$

$$\Rightarrow 8^{\log x} + 3^{1-\log x}\cdot(3\cdot 8)^{1+\log x} = 73 \Rightarrow 8^{\log x} + 3^{1-\log x}\cdot 3^{1+\log x}\cdot 8^{1+\log x} = 73 \Rightarrow$$

$$\Rightarrow 8^{\log x} + 3^{1-\log x+1+\log x}\cdot 8^{1+\log x} = 73 \Rightarrow 8^{\log x} + 3^2\cdot 8^{1+\log x} = 73 \Rightarrow$$

$$\Rightarrow 8^{\log x} + 9\cdot 8\cdot 8^{\log x} = 73 \Rightarrow 8^{\log x} + 72\cdot 8^{\log x} = 73 \Rightarrow 73\cdot 8^{\log x} = 73 \ /:73 \Rightarrow$$

$$\Rightarrow 8^{\log x} = 1 \Rightarrow 8^{\log x} = 8^0 \Rightarrow \log x = 0 \Rightarrow x = 1.$$

Vježba 124

Riješite jednadžbu: $5^{\log x} + 3^{1-\log x}\cdot 15^{\log(10\cdot x)} = 46.$

Rezultat: x = 1.

Zadatak 125 (Anamarija, maturantica TUPŠ-a)

Izračunajte: $\log_3 \sqrt[4]{3} + \log_4\left(\log_{16} 256\right).$

Rješenje 125

Ponovimo!

$$\log_b \sqrt[n]{a} = \frac{1}{n}\cdot\log_b a \quad , \quad \log_b b = 1 \quad , \quad \log_b a^n = n\cdot\log_b a \quad , \quad \log_{b^n} a = \frac{1}{n}\cdot\log_b a.$$

$$\log_3 \sqrt[4]{3} + \log_4\left(\log_{16} 256\right) = \frac{1}{4}\cdot\log_3 3 + \log_4\left(\log_{16} 16^2\right) = \frac{1}{4}\cdot 1 + \log_4\left(2\cdot\log_{16} 16\right) = \frac{1}{4} + \log_4(2\cdot 1) =$$

$$= \frac{1}{4} + \log_4 2 = \frac{1}{4} + \log_{2^2} 2 = \frac{1}{4} + \frac{1}{2}\cdot\log_2 2 = \frac{1}{4} + \frac{1}{2}\cdot 1 = \frac{1}{4} + \frac{1}{2} = \frac{1+2}{4} = \frac{3}{4}.$$

Vježba 125

Izračunajte: $\log_3 \sqrt{3} + \log_4\left(\log_{16} 256\right).$

Rezultat: 1.

Zadatak 126 (Jan, Luka, Zoran, maturanti gimnazije)

Ako je $\log_6 2 = a$, koliko je $\log_6 27$?

Rješenje 126

Ponovimo!

$$\log_b a^n = n \cdot \log_b a \quad , \quad \log_b \frac{a}{c} = \log_b a - \log_b c \quad , \quad \log_b a = \frac{\log_c a}{\log_c b} \quad , \quad \log_b a = \frac{1}{\log_a b}$$

$$\log_b b = 1 \quad , \quad \log_b b^n = n.$$

1.inačica

$$\log_6 27 = \log_6 3^3 = 3 \cdot \log_6 3 = 3 \cdot \log_6 \frac{6}{2} = 3 \cdot \left(\log_6 6 - \log_6 2\right) = 3 \cdot (1 - a) = 3 - 3 \cdot a.$$

2.inačica

$$\log_6 27 = \frac{\log_2 27}{\log_2 6} = \log_2 27 \cdot \frac{1}{\log_2 6} = \log_2 27 \cdot \log_6 2 = \log_2 3^3 \cdot a = 3 \cdot \log_2 3 \cdot a = 3 \cdot a \cdot \log_2 3 =$$

$$= 3 \cdot a \cdot \log_2 \frac{6}{2} = 3 \cdot a \cdot \left(\log_2 6 - \log_2 2\right) = 3 \cdot a \cdot \left(\frac{1}{\log_6 2} - 1\right) = 3 \cdot a \cdot \left(\frac{1}{a} - 1\right) = 3 - 3 \cdot a.$$

Vježba 126

Ako je $\log_6 2 = a$, koliko je $\log_6 9$?

Rezultat: $2 - 2 \cdot a.$

Zadatak 127 (Jan, Zoran, Luka, maturanti gimnazije, Tanja, maturantica ekonomske škole)

Nađite inverznu funkciju funkcije $f(x) = \log_2 x + \log_4 x.$

Rješenje 127

Ponovimo!

$$\log_{b^n} a = \frac{1}{n} \cdot \log_b a \quad , \quad y = \log_b x \Rightarrow x = b^y \quad , \quad \left(a^n\right)^m = a^{n \cdot m} \quad , \quad b^{\log_b a} = a.$$

1.inačica

$$y = \log_2 x + \log_4 x \Rightarrow \begin{bmatrix} \text{zamijenimo } x \text{ i } y \\ x \leftrightarrow y \end{bmatrix} \Rightarrow x = \log_2 y + \log_4 y \Rightarrow x = \log_2 y + \log_{2^2} y \Rightarrow$$

$$\Rightarrow x = \log_2 y + \frac{1}{2} \cdot \log_2 y \Rightarrow x = \frac{3}{2} \cdot \log_2 y \Rightarrow \begin{bmatrix} \text{izračunamo} \\ y \end{bmatrix} \Rightarrow x = \frac{3}{2} \cdot \log_2 y \; / \cdot \frac{2}{3} \Rightarrow \frac{2}{3} \cdot x = \log_2 y \Rightarrow$$

$$\Rightarrow \log_2 y = \frac{2}{3} \cdot x \Rightarrow y = 2^{\frac{2}{3} \cdot x} \Rightarrow y = \left(2^2\right)^{\frac{x}{3}} \Rightarrow y = 4^{\frac{x}{3}} \Rightarrow f^{-1}(x) = 4^{\frac{x}{3}}.$$

2.inačica

$$y = \log_2 x + \log_4 x \Rightarrow y = \log_2 x + \log_{2^2} x \Rightarrow y = \log_2 x + \frac{1}{2} \cdot \log_2 x \Rightarrow y = \frac{3}{2} \cdot \log_2 x \Rightarrow \begin{bmatrix} \text{izračunamo} \\ x \end{bmatrix} \Rightarrow$$

$$\Rightarrow y = \frac{3}{2} \cdot \log_2 x \; / \cdot \frac{2}{3} \Rightarrow \frac{2}{3} \cdot y = \log_2 x \Rightarrow \log_2 x = \frac{2}{3} \cdot y \Rightarrow x = 2^{\frac{2}{3} \cdot y} \Rightarrow x = \left(2^2\right)^{\frac{y}{3}} \Rightarrow x = 4^{\frac{y}{3}} \quad f^{-1}(x) = 4^{\frac{x}{3}}.$$

Vježba 127

Nađite inverznu funkciju funkcije $f(x) = \log_3 x + \log_9 x.$

Rezultat: $f^{-1}(x) = 9^{\frac{x}{3}}.$

Zadatak 128 (Jan, Zoran, Luka, maturanti gimnazije, Tanja, maturantica ekonomske škole)

Koliko realnih korijena (rješenja) ima jednadžba $(2\cdot x+1)^{3-x}=(2\cdot x+1)^{7-x}$.

Rješenje 128

Uočimo da se nepoznanica x nalazi u bazi i eksponentu svake potencije zadane jednadžbe. Zato vrijedi:

- potencije kojima je baza 0 jednake su za bilo koji eksponent (realan broj) različit od nule:

$$(2\cdot x+1)^{3-x}=(2\cdot x+1)^{7-x}\Rightarrow\begin{bmatrix}0^n=0^m\\ n,\ m\in R\setminus\{0\}\end{bmatrix}\Rightarrow 2\cdot x+1=0\Rightarrow 2\cdot x=-1\ /:2\Rightarrow x_1=-\frac{1}{2}.\ \text{rješenje}$$

- potencije kojima je baza 1 jednake su za bilo koji eksponent (realan broj):

$$(2\cdot x+1)^{3-x}=(2\cdot x+1)^{7-x}\Rightarrow\begin{bmatrix}1^n=1^m\\ n,\ m\in R\end{bmatrix}\Rightarrow 2\cdot x+1=1\Rightarrow 2\cdot x=1-1\Rightarrow 2\cdot x=0\ /:2\Rightarrow x_2=0.\ \text{rješenje}$$

- potencije kojima je baza – 1 jednake su za bilo koji paran eksponent (realan broj):

$$(2\cdot x+1)^{3-x}=(2\cdot x+1)^{7-x}\Rightarrow\begin{bmatrix}(-1)^n=(-1)^m\\ n,\ m\in R\end{bmatrix}\Rightarrow 2\cdot x+1=-1\Rightarrow 2\cdot x=-1-1\Rightarrow$$

$$\Rightarrow 2\cdot x=-2\ /:2\Rightarrow x_3=-1.\ \text{rješenje}$$

Jednadžba ima 3 realna korijena (rješenja).

Vježba 128

Koliko realnih korijena (rješenja) ima jednadžba $(3\cdot x+1)^{3-x}=(3\cdot x+1)^{7-x}$.

Rezultat: Jednadžba ima 3 realna korijena (rješenja).

Zadatak 129 (Jan, Zoran, Luka, maturanti gimnazije, Tanja, maturantica ekonomske škole)

Koliki je zbroj svih rješenja jednadžbe $\log\frac{1}{x}\cdot\log x-\log\left(100\cdot x^3\right)=0$?

Rješenje 129

Ponovimo!

$$\log a^n=\frac{1}{n}\cdot\log a\quad,\quad\log(a\cdot b)=\log a+\log b\quad,\quad a^{-n}=\frac{1}{a^n}.$$

$$\log\frac{1}{x}\cdot\log x-\log\left(100\cdot x^3\right)=0\Rightarrow\log x^{-1}\cdot\log x-\left(\log 100+\log x^3\right)=0\Rightarrow-\log x\cdot\log x-(2+3\cdot\log x)=0\Rightarrow$$

$$\Rightarrow-\log^2 x-2-3\cdot\log x=0\Rightarrow-\log^2 x-3\cdot\log x-2=0\ /\cdot(-1)\Rightarrow\log^2 x+3\cdot\log x+2=0\Rightarrow$$

$$\Rightarrow\begin{bmatrix}\text{supstitucija}\\ \log x=t\end{bmatrix}\Rightarrow t^2+3\cdot t+2=0\Rightarrow\left.\begin{array}{l}a=1\ ,\ b=3\ ,\ c=2\\ t_{1,2}=\frac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}\end{array}\right\}\Rightarrow t_{1,2}=\frac{-3\pm\sqrt{9-4\cdot1\cdot2}}{2\cdot1}\Rightarrow$$

$$\Rightarrow t_{1,2}=\frac{-3\pm\sqrt{9-8}}{2}\Rightarrow t_{1,2}=\frac{-3\pm\sqrt{1}}{2}\Rightarrow t_{1,2}=\frac{-3\pm1}{2}\Rightarrow\left.\begin{array}{l}t_1=\frac{-3+1}{2}\\ t_2=\frac{-3-1}{2}\end{array}\right\}\Rightarrow\left.\begin{array}{l}t_1=\frac{-2}{2}\\ t_2=\frac{-4}{2}\end{array}\right\}\Rightarrow\left.\begin{array}{l}t_1=-1\\ t_2=-2\end{array}\right\}.$$

Vraćamo se na supstituciju:

$$\left.\begin{array}{l}t=-1\\ \log x=t\end{array}\right\}\Rightarrow\log x=-1\Rightarrow x_1=10^{-1}\Rightarrow x_1=0.1.$$

$$\left.\begin{array}{l}t=-2\\ \log x=t\end{array}\right\}\Rightarrow\log x=-2\Rightarrow x_2=10^{-2}\Rightarrow x_2=0.01.$$

Zbroj rješenja iznosi:

$$x_1 + x_2 = 0.1 + 0.01 \Rightarrow x_1 + x_2 = 0.11.$$

Vježba 129

Koliki je umnožak svih rješenja jednadžbe $\log\frac{1}{x}\cdot\log x - \log\left(100\cdot x^3\right) = 0$?

Rezultat: 0.001.

Zadatak 130 (Tanja, ekonomska škola)

Riješite nejednadžbu: $\log_x 3\cdot\log_3(3\cdot x - 1) > 1.$

Rješenje 130

Ponovimo!

$$\log_b a = c \Rightarrow b^c = a \;,\; b > 0 \;,\; b \neq 1 \quad , \quad \log_b a = \frac{1}{\log_a b} \quad , \quad \log_b a = \frac{\log_c a}{\log_c b} \quad , \quad \log_b b = 1$$

$$\log_b f(x) > \log_b g(x) \;,\; 0 < b < 1 \Rightarrow f(x) < g(x) \quad , \quad \log_b f(x) > \log_b g(x) \;,\; b > 1 \Rightarrow f(x) > g(x).$$

$$\log_x 3\cdot\log_3(3\cdot x - 1) > 1 \Rightarrow \frac{1}{\log_3 x}\cdot\log_3(3\cdot x - 1) > 1 \Rightarrow \frac{\log_3(3\cdot x - 1)}{\log_3 x} > 1 \Rightarrow \log_x(3\cdot x - 1) > 1 \Rightarrow$$

$$\Rightarrow \log_x(3\cdot x - 1) > \log_x x.$$

1.slučaj

$$\left.\begin{matrix} 0 < x < 1 \\ 3\cdot x - 1 > 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 0 < x < 1 \\ 3\cdot x > 1 \;/:3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 0 < x < 1 \\ x > \frac{1}{3} \end{matrix}\right\} \Rightarrow x \in \left\langle \frac{1}{3}, 1 \right\rangle.$$

Za tako određeni x vrijedi:

$$\log_x(3\cdot x - 1) > \log_x x \Rightarrow 3\cdot x - 1 < x \Rightarrow 3\cdot x - x < 1 \Rightarrow 2\cdot x < 1 \;/:2 \Rightarrow x < \frac{1}{2} \Rightarrow \left.\begin{matrix} x \in \left\langle \frac{1}{3}, 1 \right\rangle \\ x < \frac{1}{2} \end{matrix}\right\} \Rightarrow x \in \left\langle \frac{1}{3}, \frac{1}{2} \right\rangle.$$

2.slučaj

$$\left.\begin{matrix} x > 1 \\ 3\cdot x - 1 > 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x > 1 \\ 3\cdot x > 1 \;/:3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x > 1 \\ x > \frac{1}{3} \end{matrix}\right\} \Rightarrow x \in \left\langle 1, +\infty \right\rangle.$$

Za tako određeni x vrijedi:

$$\log_x(3\cdot x - 1) > \log_x x \Rightarrow 3\cdot x - 1 > x \Rightarrow 3\cdot x - x > 1 \Rightarrow 2\cdot x > 1 \;/:2 \Rightarrow x > \frac{1}{2} \Rightarrow \left.\begin{matrix} x \in \left\langle 1, +\infty \right\rangle \\ x > \frac{1}{2} \end{matrix}\right\} \Rightarrow x \in \left\langle 1, +\infty \right\rangle.$$

Konačno rješenje nejednadžbe glasi:

$$x \in \left\langle \frac{1}{3}, \frac{1}{2} \right\rangle \cup \left\langle 1, +\infty \right\rangle.$$

Vježba 130

Riješite nejednadžbu: $\log_x 2\cdot\log_2(4\cdot x) > 1.$

Rezultat: $x \in \left\langle 1, +\infty \right\rangle.$

Zadatak 131 (Iva, ekonomska škola)

Riješite jednadžbu: $\frac{1}{9\cdot 5^{1+x} + 5^{2+x} + 5^{3+x}} = \frac{1}{3^{2+x} + 3^{3+x} + 3^{4+x}}.$

Rješenje 131

Ponovimo!

$$a^{x+y}=a^x\cdot a^y \quad , \quad a^1=a \quad , \quad \frac{a^x}{b^x}=\left(\frac{a}{b}\right)^x \quad , \quad \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n \quad , \quad a^{f(x)}=a^{g(x)} \Rightarrow f(x)=g(x).$$

$$\frac{1}{9\cdot 5^{1+x}+5^{2+x}+5^{3+x}}=\frac{1}{3^{2+x}+3^{3+x}+3^{4+x}} \Rightarrow 3^{2+x}+3^{3+x}+3^{4+x}=9\cdot 5^{1+x}+5^{2+x}+5^{3+x} \Rightarrow$$

$$\Rightarrow 3^2\cdot 3^x+3^3\cdot 3^x+3^4\cdot 3^x=9\cdot 5^1\cdot 5^x+5^2\cdot 5^x+5^3\cdot 5^x \Rightarrow 3^x\cdot\left(3^2+3^3+3^4\right)=5^x\cdot\left(9\cdot 5^1+5^2+5^3\right) \Rightarrow$$

$$\Rightarrow 3^x\cdot(9+27+81)=5^x\cdot(45+25+125) \Rightarrow 3^x\cdot 117=5^x\cdot 195 \;/\cdot\frac{1}{117\cdot 5^x} \Rightarrow \frac{3^x}{5^x}=\frac{195}{117} \Rightarrow$$

$$\Rightarrow \left(\frac{3}{5}\right)^x=\frac{3\cdot 5\cdot 13}{3\cdot 3\cdot 13} \Rightarrow \left(\frac{3}{5}\right)^x=\frac{5}{3} \Rightarrow \left(\frac{3}{5}\right)^x=\left(\frac{3}{5}\right)^{-1} \Rightarrow x=-1.$$

Vježba 131

Riješite jednadžbu: $\frac{1}{5\cdot 2^{x+1}+2^{x+2}}=\frac{1}{3^{x+2}+3^{x-1}}$.

Rezultat: x = 1.

Zadatak 132 (Edita, maturantica)

Nađi realno rješenje jednadžbe: $\frac{1}{7^{1+x}-3\cdot 2^{3+x}}=\frac{1}{5\cdot 7^x+2^{-1+x}}$.

Rješenje 132

Ponovimo!

$$\frac{1}{a}=\frac{1}{b} \Rightarrow a=b \quad , \quad a^x\cdot b^x=(a\cdot b)^x \quad , \quad a^1=a \quad , \quad a^0=1 \quad , \quad a^{-n}=\frac{1}{a^n}$$

$$a^x=b^x \Rightarrow \frac{a^x}{b^x}=1 \Rightarrow \left(\frac{a}{b}\right)^x=\left(\frac{a}{b}\right)^0 \Rightarrow x=0.$$

U zadanoj jednadžbi izjednačimo nazivnike:

$$\frac{1}{7^{1+x}-3\cdot 2^{3+x}}=\frac{1}{5\cdot 7^x+2^{-1+x}} \Rightarrow 7^{1+x}-3\cdot 2^{3+x}=5\cdot 7^x+2^{-1+x} \Rightarrow$$

$$\Rightarrow 7^1\cdot 7^x-3\cdot 2^3\cdot 2^x=5\cdot 7^x+2^{-1}\cdot 2^x \Rightarrow 7\cdot 7^x-24\cdot 2^x=5\cdot 7^x+\frac{1}{2}\cdot 2^x \Rightarrow$$

$$\Rightarrow 7\cdot 7^x-5\cdot 7^x=\frac{1}{2}\cdot 2^x+24\cdot 2^x \Rightarrow 7^x\cdot(7-5)=2^x\cdot\left(\frac{1}{2}+24\right) \Rightarrow 7^x\cdot 2=2^x\cdot\frac{49}{2} \;/\cdot\frac{1}{2\cdot 49} \Rightarrow$$

$$\Rightarrow \frac{7^x}{49}=\frac{2^x}{4} \Rightarrow \frac{7^x}{7^2}=\frac{2^x}{2^2} \Rightarrow 7^{x-2}=2^{x-2} \;/\cdot\frac{1}{2^{x-2}} \Rightarrow \frac{7^{x-2}}{2^{x-2}}=1 \Rightarrow \left(\frac{7}{2}\right)^{x-2}=1 \Rightarrow$$

$$\Rightarrow \left(\frac{7}{2}\right)^{x-2}=\left(\frac{7}{2}\right)^0 \Rightarrow x-2=0 \Rightarrow x=2.$$

Vježba 132

Nađi realno rješenje jednadžbe: $\frac{1}{7^{2+x}}=\frac{1}{24\cdot 7^x+25\cdot 2^x}$.

Rezultat: 0.

Zadatak 133 (Anita, farmaceutska škola)

Riješite nejednadžbu: $3^{2\cdot x-1}+3^{2\cdot x-2}-3^{2\cdot x-4}\le 315.$

Rješenje 133

Ponovimo!

$$a^{x} \cdot a^{y} = a^{x+y} \quad , \quad a^{-n} = \frac{1}{a^{n}} \quad , \quad a^{f(x)} = a^{g(x)} \Rightarrow f(x) = g(x)$$

$$a^{f(x)} \le a^{g(x)} \ , \ a > 1 \Rightarrow f(x) \le g(x).$$

1.inačica

$$3^{2\cdot x-1} + 3^{2\cdot x-2} - 3^{2\cdot x-4} \le 315 \Rightarrow 3^{2\cdot x} \cdot 3^{-1} + 3^{2\cdot x} \cdot 3^{-2} - 3^{2\cdot x} \cdot 3^{-4} \le 315 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x} \cdot \left(3^{-1} + 3^{-2} - 3^{-4}\right) \le 315 \Rightarrow 3^{2\cdot x} \cdot \left(\frac{1}{3} + \frac{1}{3^{2}} - \frac{1}{3^{4}}\right) \le 315 \Rightarrow 3^{2\cdot x} \cdot \left(\frac{1}{3} + \frac{1}{9} - \frac{1}{81}\right) \le 315 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x} \cdot \frac{27+9-1}{81} \le 315 \Rightarrow 3^{2\cdot x} \cdot \frac{35}{81} \le 315 \ / \cdot \frac{81}{35} \Rightarrow 3^{2\cdot x} \le 729 \Rightarrow 3^{2\cdot x} \le 3^{6} \Rightarrow$$

$$\Rightarrow 2 \cdot x \le 6 \ /:2 \Rightarrow x \le 3 \Rightarrow x \in \langle -\infty, 3].$$

2.inačica

$$3^{2\cdot x-1} + 3^{2\cdot x-2} - 3^{2\cdot x-4} \le 315 \Rightarrow 3^{2\cdot x-4} \cdot \left(3^{3} + 3^{2} - 1\right) \le 315 \Rightarrow 3^{2\cdot x-4} \cdot (27+9-1) \le 315 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x-4} \cdot 35 \le 315 \ /:35 \Rightarrow 3^{2\cdot x-4} \le 9 \Rightarrow 3^{2\cdot x-4} \le 3^{2} \Rightarrow 2 \cdot x - 4 \le 2 \Rightarrow$$

$$\Rightarrow 2 \cdot x \le 6 \ /:2 \Rightarrow x \le 3 \Rightarrow x \in \langle -\infty, 3].$$

Vježba 133

Riješite nejednadžbu: $3^{2\cdot x-1} + 3^{2\cdot x-2} - 3^{2\cdot x-4} \le 35.$

Rezultat: $x \in \langle -\infty, 2].$

Zadatak 134 (Rea, gimnazija)

Riješite jednadžbu: $50^{\log x} \cdot 160^{\log x} = 400.$

Rješenje 134

Ponovimo!

$$a^{x} \cdot b^{x} = (a \cdot b)^{x} \quad , \quad \left(a^{x}\right)^{y} = a^{x\cdot y} \quad , \quad a^{f(x)} = a^{g(x)} \Rightarrow f(x) = g(x) \quad , \quad \log_{b} a = c \Leftrightarrow b^{c} = a$$

$$a^{\frac{x}{y}} = \sqrt[y]{a^{x}} \quad , \quad \log a^{x} = x \cdot \log a \quad , \quad \log_{b} b^{x} = x \quad , \quad \log f(x) = \log g(x) \Rightarrow f(x) = g(x).$$

1.inačica

$$50^{\log x} \cdot 160^{\log x} = 400 \Rightarrow (50 \cdot 160)^{\log x} = 400 \Rightarrow 8000^{\log x} = 400 \Rightarrow \left(20^{3}\right)^{\log x} = 20^{2} \Rightarrow$$

$$\Rightarrow 20^{3\cdot \log x} = 20^{2} \Rightarrow 3 \cdot \log x = 2 \ /:3 \Rightarrow \log x = \frac{2}{3} \Rightarrow x = 10^{\frac{2}{3}} \Rightarrow x = \sqrt[3]{10^{2}} \Rightarrow x = \sqrt[3]{100}.$$

2.inačica

$$50^{\log x} \cdot 160^{\log x} = 400 \Rightarrow (50 \cdot 160)^{\log x} = 400 \Rightarrow 8000^{\log x} = 400 \Rightarrow \left(20^{3}\right)^{\log x} = 20^{2} \Rightarrow$$

$$\Rightarrow 20^{3\cdot \log x} = 20^{2} \Rightarrow 3 \cdot \log x = 2 \Rightarrow \log x^{3} = \log 100 \Rightarrow x^{3} = 100 \ / \sqrt[3]{\ } \Rightarrow x = \sqrt[3]{100}.$$

Vježba 134

Riješite jednadžbu: $25^{\log x} \cdot 320^{\log x} = 400.$

Rezultat: $x = \sqrt[3]{100}.$

Zadatak 135 (Radovan, tehnička škola)

Riješite nejednadžbu: $\log_{\frac{1}{2}} \sin x < 0.$

Rješenje 135

Ponovimo!

$$\log_b a = c \;,\; a > 0 \;,\; b > 0 \;,\; b \neq 1 \Leftrightarrow b^c = a \quad , \quad \log_b 1 = 0 \quad , \quad \log_b f(x) < \log_b g(x) \;,\; 0 < b < 1 \Rightarrow f(x) > g(x).$$

$$\log_{\frac{1}{2}} \sin x < 0 \Rightarrow \begin{bmatrix} \text{diskusija} \\ \sin x > 0 \end{bmatrix} \Rightarrow \log_{\frac{1}{2}} \sin x < \log_{\frac{1}{2}} 1 \Rightarrow \sin x > 1 \text{ nema rješenja jer je } |\sin x| \leq 1.$$

Vježba 135

Riješite nejednadžbu: $\log_{\frac{1}{3}} \cos x < 0.$

Rezultat: Nema rješenja.

Zadatak 136 (Boris, maturant gimnazije)

Ako je $\log a + 3 \cdot \log b = 2$ i $\log a - \log b = 1$, koliko je $a \cdot b$?

Rješenje 136

Ponovimo!

$$\log x^n = n \cdot \log x \quad , \quad \log x - \log y = \log \frac{x}{y} \quad , \quad \log x + \log y = \log(x \cdot y) \quad , \quad \log_x x^n = n \quad , \quad x^n \cdot y^n = (x \cdot y)^n$$

$$\log_y x = z \Leftrightarrow y^z = x \quad , \quad \sqrt[n]{x^m} = x^{\frac{m}{n}} \quad , \quad \sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y} \quad , \quad \log f(x) = \log g(x) \Rightarrow f(x) = g(x).$$

1.inačica

$$\left.\begin{array}{l} \log a + 3 \cdot \log b = 2 \\ \log a - \log b = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \log a + \log b^3 = \log 100 \\ \log \frac{a}{b} = \log 10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \log\left(a \cdot b^3\right) = \log 100 \\ \log \frac{a}{b} = \log 10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a \cdot b^3 = 100 \\ \frac{a}{b} = 10 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{pomnožimo} \\ \text{jednakosti} \end{bmatrix} \Rightarrow a \cdot b^3 \cdot \frac{a}{b} = 100 \cdot 10 \Rightarrow a^2 \cdot b^2 = 1000 \Rightarrow (a \cdot b)^2 = 1000 \;/\sqrt{\;} \Rightarrow$$

$$\Rightarrow a \cdot b = \sqrt{1000} \Rightarrow \begin{bmatrix} \text{djelomično} \\ \text{korjenovanje} \end{bmatrix} \Rightarrow a \cdot b = \sqrt{100 \cdot 10} \Rightarrow a \cdot b = 10 \cdot \sqrt{10}.$$

2.inačica

$$\left.\begin{array}{l} \log a + 3 \cdot \log b = 2 \\ \log a - \log b = 1 \end{array}\right\} \Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow \left.\begin{array}{l} \log a + 3 \cdot \log b = 2 \\ \log a - \log b = 1 \;/\cdot 3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \log a + 3 \cdot \log b = 2 \\ 3 \cdot \log a - 3 \cdot \log b = 3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 4 \cdot \log a = 5 \Rightarrow \log a = \frac{5}{4} \Rightarrow a = 10^{\frac{5}{4}} \Rightarrow a = \sqrt[4]{10^5} \Rightarrow \left.\begin{array}{l} a = 10^{\frac{5}{4}} \\ \log a - \log b = 1 \end{array}\right\} \Rightarrow \log 10^{\frac{5}{4}} - \log b = 1 \Rightarrow$$

$$\Rightarrow \frac{5}{4} - \log b = 1 \Rightarrow -\log b = 1 - \frac{5}{4} \Rightarrow -\log b = -\frac{1}{4} \;/\cdot(-1) \Rightarrow \log b = \frac{1}{4} \Rightarrow b = 10^{\frac{1}{4}} \Rightarrow b = \sqrt[4]{10}.$$

Umnožak a · b iznosi:

$$a \cdot b = \sqrt[4]{10^5} \cdot \sqrt[4]{10} \Rightarrow a \cdot b = \sqrt[4]{10^5 \cdot 10} \Rightarrow a \cdot b = \sqrt[4]{10^6} \Rightarrow a \cdot b = \sqrt{10^3} \Rightarrow \begin{bmatrix} \text{djelomično} \\ \text{korjenovanje} \end{bmatrix} \Rightarrow$$

$$\Rightarrow a \cdot b = \sqrt{10^2 \cdot 10} \Rightarrow a \cdot b = 10 \cdot \sqrt{10}.$$

Vježba 136

Ako je $\log a + 3 \cdot \log b = 6$ i $\log a - \log b = 2$, koliko je $a \cdot b$?

Rezultat: 10 000.

Zadatak 137 (Ivan, gimnazija)

Dokaži da za a > 0, b > 0, a ≠ 1, b ≠ 1 vrijedi:

$$\frac{1}{\log_a b}+\frac{1}{\log_{a^2} b}+\frac{1}{\log_{a^3} b}+\ ...\ +\frac{1}{\log_{a^n} b}=\frac{n\cdot(n+1)}{2\cdot\log_a b}.$$

Rješenje 137

Ponovimo!

$$\log_y x=\frac{1}{\log_x y}\quad,\quad \log_y x^n=n\cdot\log_y x\quad,\quad 1+2+3+...+n=\frac{n\cdot(n+1)}{2}.$$

$$\frac{1}{\log_a b}+\frac{1}{\log_{a^2} b}+\frac{1}{\log_{a^3} b}+\ ...\ +\frac{1}{\log_{a^n} b}=\log_b a+\log_b a^2+\log_b a^3+\ ...\ +\log_b a^n=$$

$$=\log_b a+2\cdot\log_b a+3\cdot\log_b a+\ ...\ +n\cdot\log_b a=\log_b a\cdot(1+2+3+...+n)=\log_b a\cdot\frac{n\cdot(n+1)}{2}=$$

$$=\frac{n\cdot(n+1)}{2}\cdot\frac{1}{\log_a b}=\frac{n\cdot(n+1)}{2\cdot\log_a b}.$$

Vježba 137

Dokaži da za a > 0, b > 0, a ≠ 1, b ≠ 1 vrijedi:

$$\frac{1}{\log_a b}+\frac{1}{\log_{a^2} b}+\frac{1}{\log_{a^3} b}+\frac{1}{\log_{a^4} b}+\frac{1}{\log_{a^5} b}=15\cdot\log_b a.$$

Rezultat: Dokaz analogan.

Zadatak 138 (Ivan, gimnazija)

Dokaži: $\frac{\log_a x}{\log_{ab} x}=1+\log_a b,$ za sve pozitivne brojeve a, b i x.

Rješenje 138

Ponovimo!

$$\log_y x=\frac{1}{\log_x y}\quad,\quad \log_y x=\frac{\log_z x}{\log_z y}\quad,\quad \log_b(x\cdot y)=\log_b x+\log_b y\quad,\quad \log_x x=1.$$

1.inačica

$$\frac{\log_a x}{\log_{ab} x}=\frac{\log_x(a\cdot b)}{\log_x a}=\frac{\log_x a+\log_x b}{\log_x a}=\frac{\log_x a}{\log_x a}+\frac{\log_x b}{\log_x a}=1+\frac{\log_x b}{\log_x a}=1+\log_a b.$$

2.inačica

$$\frac{\log_a x}{\log_{ab} x}=\frac{\log_x(a\cdot b)}{\log_x a}=\log_a(a\cdot b)=\log_a a+\log_a b=1+\log_a b.$$

Vježba 138

Dokaži: $\frac{\log_2 5}{\log_6 5}=1+\log_2 3,$ za sve pozitivne brojeve a, b i x.

Rezultat: Dokaz analogan.

Zadatak 139 (Leon, gimnazija)

Riješite sustav: $\begin{cases}2^{x+1}\cdot y=24\\ 3^{x-1}\cdot y^2=27.\end{cases}$

Rješenje 139

Ponovimo!

$$a^n\cdot a^m=a^{n+m}\quad,\quad \left(a^n\right)^m=a^{n\cdot m}\quad,\quad \frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n\quad,\quad a^0=1\quad,\quad a^{f(x)}=a^{g(x)}\ \Rightarrow\ f(x)=g(x).$$

$$\left.\begin{array}{l}2^{x+1}\cdot y=24\\3^{x-1}\cdot y^{2}=27\end{array}\right\}\Rightarrow\left.\begin{array}{l}2^{x}\cdot 2\cdot y=24\ /:2\\3^{x}\cdot 3^{-1}\cdot y^{2}=27\ /\cdot 3\end{array}\right\}\Rightarrow\left.\begin{array}{l}2^{x}\cdot y=12\\3^{x}\cdot 3^{-1}\cdot 3\cdot y^{2}=81\end{array}\right\}\Rightarrow\left.\begin{array}{l}2^{x}\cdot y=12\\3^{x}\cdot 3^{0}\cdot y^{2}=81\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}2^{x}\cdot y=12\ /^{2}\\3^{x}\cdot y^{2}=81\end{array}\right\}\Rightarrow\left.\begin{array}{l}\left(2^{x}\right)^{2}\cdot y^{2}=144\\3^{x}\cdot y^{2}=81\end{array}\right\}\Rightarrow\left.\begin{array}{l}\left(2^{2}\right)^{x}\cdot y^{2}=144\\3^{x}\cdot y^{2}=81\end{array}\right\}\Rightarrow\left.\begin{array}{l}4^{x}\cdot y^{2}=144\\3^{x}\cdot y^{2}=81\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{podijelimo}\\\text{jednadžbe}\end{array}\right]\Rightarrow$$

$$\Rightarrow\frac{4^{x}\cdot y^{2}}{3^{x}\cdot y^{2}}=\frac{144}{81}\Rightarrow\frac{4^{x}}{3^{x}}=\frac{144}{81}\Rightarrow\frac{4^{x}}{3^{x}}=\frac{16}{9}\Rightarrow\left(\frac{4}{3}\right)^{x}=\left(\frac{4}{3}\right)^{2}\Rightarrow x=2\Rightarrow\left.\begin{array}{l}x=2\\2^{x}\cdot y=12\end{array}\right\}\Rightarrow$$

$$\Rightarrow 2^{2}\cdot y=12\Rightarrow 4\cdot y=12\ /:4\Rightarrow y=3.$$

Rješenje sustava iznosi:

$$(x,\ y)=(2,\ 3).$$

Vježba 139

Riješite sustav: $\begin{cases}2^{x+1}\cdot y=8\\3^{x-1}\cdot y^{2}=4.\end{cases}$

Rezultat: $(x,\ y)=(1,\ 2).$

Zadatak 140 (Ivana, maturantica)

Riješite jednadžbu: $2\cdot 2^{2\cdot x}+4^{x+2}-2\cdot 4^{x-1}=35.$

Rješenje 140

Ponovimo!

$$\left(a^{n}\right)^{m}=a^{n\cdot m}\quad,\quad a^{n}\cdot a^{m}=a^{n+m}\quad,\quad a^{-n}=\frac{1}{a^{n}},\quad a^{1}=a\quad,\quad a^{f(x)}=a^{g(x)}\Rightarrow f(x)=g(x).$$

$$2\cdot 2^{2\cdot x}+4^{x+2}-2\cdot 4^{x-1}=35\Rightarrow 2\cdot 2^{2\cdot x}+\left(2^{2}\right)^{x+2}-2\cdot\left(2^{2}\right)^{x-1}=35\Rightarrow$$

$$\Rightarrow 2\cdot 2^{2\cdot x}+2^{2\cdot x+4}-2\cdot 2^{2\cdot x-2}=35\Rightarrow 2\cdot 2^{2\cdot x}+2^{2\cdot x}\cdot 2^{4}-2\cdot 2^{2\cdot x}\cdot 2^{-2}=35\Rightarrow$$

$$\Rightarrow 2^{2\cdot x}\cdot\left(2+2^{4}-2\cdot 2^{-2}\right)=35\Rightarrow 2^{2\cdot x}\cdot\left(2+2^{4}-2^{-1}\right)=35\Rightarrow 2^{2\cdot x}\cdot\left(2+16-\frac{1}{2}\right)=35\Rightarrow$$

$$\Rightarrow 2^{2\cdot x}\cdot\left(18-\frac{1}{2}\right)=35\Rightarrow 2^{2\cdot x}\cdot\frac{35}{2}=35\ /\cdot\frac{2}{35}\Rightarrow 2^{2\cdot x}=2\Rightarrow 2^{2\cdot x}=2^{1}\Rightarrow 2\cdot x=1\ /:2\Rightarrow x=\frac{1}{2}.$$

Vježba 140

Riješite jednadžbu: $2\cdot 2^{2\cdot x}+4^{x+2}=18.$

Rezultat: x = 0.